Published in *The Mathematical Gazette*, July 2018, Vol. 102, No. 554, pp. 328-330.

A surprising 3D result involving a hexagon

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In a recent posting (June 2017) on *Facebook*, the first author presented a relatively unknown theorem regarding the concurrency of lines connecting the midpoints of the opposite sides of a plane hexagon with opposite sides parallel, but not necessarily equal (see [1]). To prove this theorem, it was necessary to employ projective geometry methods as using simple Euclidean methods do not suffice. In response to this posting, John Berry (a mathematical friend for many years from York, UK) asked the question on *Facebook* whether this result would also be true in 3D space.

Obviously these lines will be concurrent for a spatial 3D hexagon with opposite sides equal **and** parallel as the concurrency follows immediately from the joint central point of (reflective) symmetry of the three parallelograms formed in space. However, the question arose whether one could construct a spatial hexagon with opposites sides parallel, but not equal.

Experimentally exploring this possibility using the dynamic 3D software *Cabri 3D* by construction and measurement (e.g. [2]), it turned out much to our surprise, that in 3D space, if a hexagon has three pairs of opposite sides parallel, then they are equal also[1]. This seemed rather counter-intuitive to us as we had expected to be able to produce a 3D analogue to a plane hexagon with opposite sides parallel, but not necessarily equal, especially since one normally has more degrees of freedom in space.

Below is now presented a Euclidean proof of this result and its point (reflective) symmetric (see footnote 2 further on) property, followed by a vector proof.

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[1] Thanks also to Kate Mackrell from Kingston, Ontario, Canada who similarly experimentally confirmed our finding with *Cabri 3D*. Readers are invited to experimentally establish the validity of the conjecture with other available software with dynamic 3D facilities.

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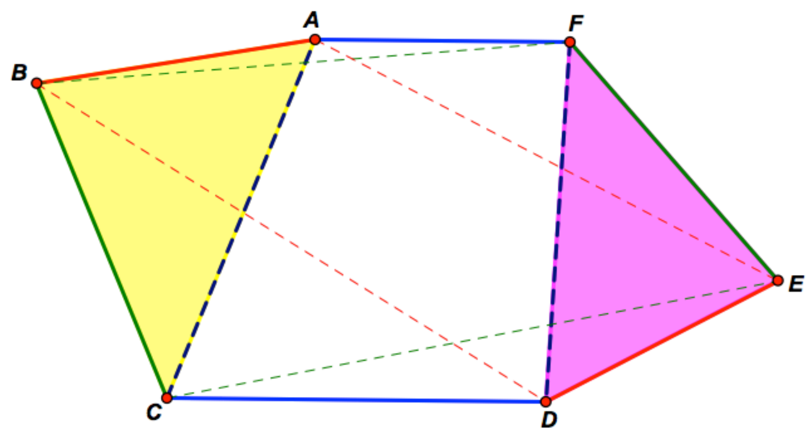

Figure 1

Proof 1

Given *ABCDEF* as a 3-D hexagon (i.e. not all six points lie in a common plane) with *AB* parallel to *DE*, *BC* parallel to *EF* and *CD* parallel to *FA* (see Figure 1 showing the hexagon in 3D). The six straight lines then lie pairwise parallel in 3 different directions in space. Since *AB* is parallel to *ED* and *BC* is parallel to *FE*, the planes *ABC* and *DEF* on which they respectively lie are also parallel to each other. Therefore *AC* must be parallel to *FD*, but since the straight lines *CD* and *FA* are given as parallel to each another, it follows that *FACD* is a parallelogram. Similarly, the quadrilaterals *ABDE* and *BCEF* are parallelograms.

Since *FACD* is a parallelogram, diagonals *AD* and *FC* have a common midpoint *M*. But *ABDE* is also a parallelogram, and hence *M* must also the midpoint of *BE*. Hence, the three diagonals of the non-planar 3D parallelo-hexagon are concurrent at *M*, it is a centre of point symmetry[2].

[2] An object has a centre of 'point symmetry' *P* if it is invariant under a *point reflection* around that point *P*.

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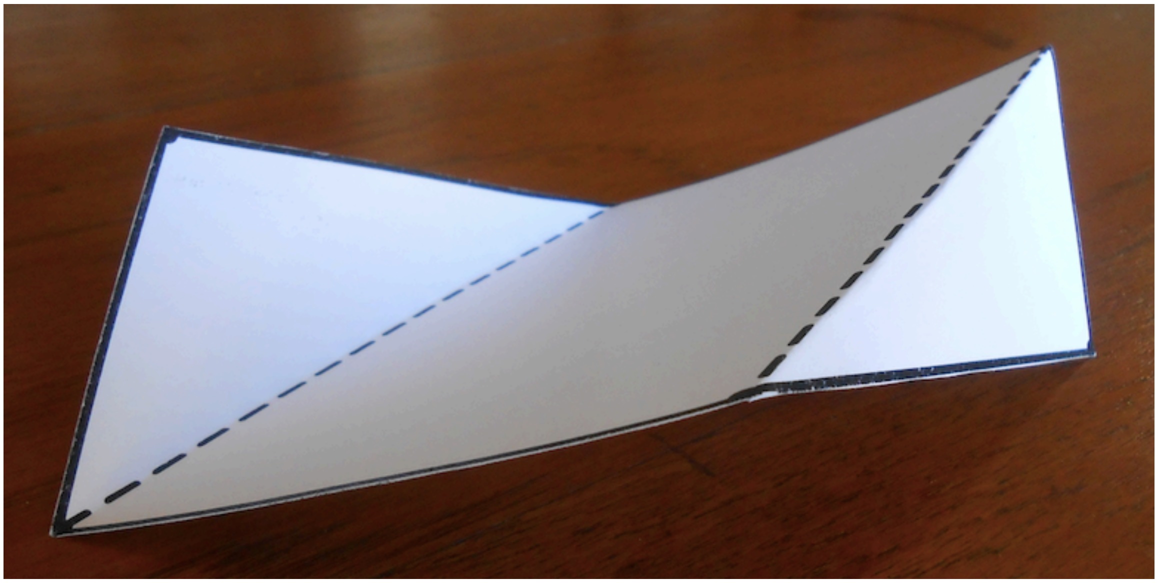

Figure 2

Note that the first proof was partly inspired by the construction of a non-planar 3D parallelo-hexagon by paper folding as shown by the photograph in Figure 2.

Proof 2

Let **A**, **B**, **C**, **D**, **E** and **F** be the position vectors of the vertices.

Because of parallelism, there are real x, y, and z non-zero with **E** - **D** = x(**B**-**A**), **F** - **E** = y(**C** - **B**) and **A** - **F** = z(**D** - **C**).

Note firstly that since the spatial hexagon is non-planar, the vectors {(**B** - **A**), (**C** - **B**), (**D** -**C**)} are linearly independent. In addition, since the spatial hexagon is closed, it follows (**B** - **A**) + (**C** - **B**) + (**D** - **C**) + x(**B** - **A**) + y(**C** - **B**) + z(**D** - **C**) = 0 (zero vector).

Thus, $x = y = z = -1$ and thus **E** - **D** = **A** - **B**, **F** - **E** = **B** - **C** and **A** - **F** = **C** – **D**, i.e. opposite sides are equal.

Let **M** be the centre of AD, that is, **M** = ½ (**A** + **D**); Then **M** = ½ (**B** + **E**) and further **M** = ½ (**C** + **F**).

A generalization for a "spatial" $2n$-gon in the n-dimensional space is now immediately obvious from this vector proof, giving a good example of a proof illustrating not only the 'explanatory' function of proof, but its 'discovery' function as well (see [3]). Clearly this proof does not work in the plane, since the requirement of linear independence does not hold.

Additionally, though there is no 3D analogue for a hexagon in the plane with opposite sides parallel, but not necessarily equal, it became apparent with the advantage of hindsight, that the 3D parallelo-hexagon was the 3D analogue of a 2D parallelogram. However, unlike the 2D parallelogram, the converse of the above result, namely that a 3D hexagon having three pairs of opposite sides equal would imply opposite sides parallel, is not true in 3D space. A counter-example is most easily constructed by considering the paper-folding example in Figure 2, and having both folded 'flaps' (*ABC* and *DEF* in Figure 1) instead folded towards the *same side* of plane *ACDF*. Though opposite sides will still be equal from such a folding, the two pairs of sides involved will no longer be correspondingly parallel.

Also note that though the 3D parallelo-hexagon has a point of (reflective) symmetry like the parallelogram, it does NOT like the parallelogram have half-turn symmetry[3] (as the 3D parallelo-hexagon can be rotated by a half-turn in say a direction perpendicular to *FACD* without mapping onto itself). It therefore also provides a good pedagogic example of showing the non-equivalence in space of the two different, but related concepts of 'point symmetry' and half-turn symmetry' (which are easy to prove equivalent in the plane, and easily illustrated by, for example, the symmetries of a parallelogram, parallelo-hexagon or a cubic function).

Concluding remarks

This little episode illustrates how some experimental exploration with 3D dynamic geometry software (as well as some paper folding) helped lead to an unanticipated conjecture (at least for us), which provides a nice problem with which to challenge high school learners or even prospective mathematics teachers, as it is relatively easy to prove. Available computing technologies is now making it much more feasible and easy to bring 3D investigations such as into the classroom.

[3] An object has a point of 'half-turn symmetry' *P* if it is invariant under a *half-turn* (rotation of 180°) around that point *P*.

References

1. M. de Villiers, Feedback: More on Hexagons with Opposite Sides Parallel. *The Mathematical Gazette*, Nov. 2006, pp. 517 -518. (An interactive, dynamic sketch illustrating the concurrency result mentioned at the start can be accessed online at: http://dynamicmathematicslearning.com/parahex.html)
2. H. Schumann. *Elementary geometry in a virtual area of action*. A book about teaching and learning with Cabri 3D on CD. (Schulgeometrie im virtuellen Handlungsraum. Ein Lehr- und Lernbuch der interaktiven Raumgeometrie mit Cabri 3D auf CD.) 2007, Hildesheim: Franzbecker.
3. M. de Villiers, The Role and Function of Proof in Mathematics. *Pythagoras*, Nov. 1990, no. 24, pp. 17 -24.

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