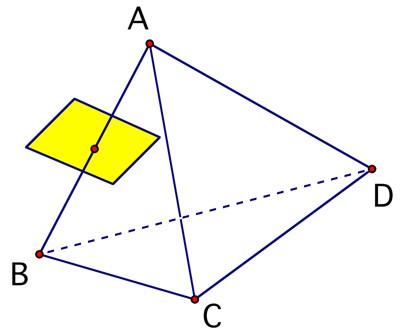
Exploring some concurrencies in the tetrahedron

Consider the corresponding analogues of perpendicular bisectors, angle bisectors, medians and altitudes of a triangle for a tetrahedron, and investigate which of these are also concurrent for a tetrahedron. If true, prove it; if not, provide a counter-example.

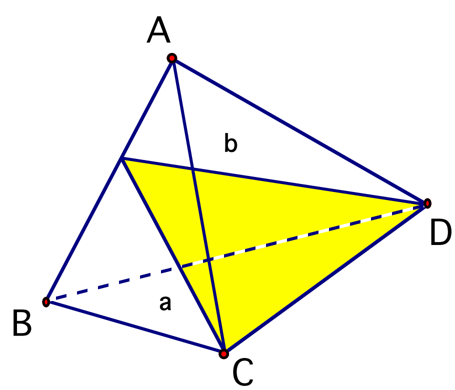
Let us start with following useful assumption (stated without proof) for 3D:
Three (non-parallel) planes meet in a point. (It is easy to see this in a rectangular room where two walls and the ceiling meet perpendicularly in a point.)

Definition 1: The *perpendicular edge bisector* is the generalisation to 3D of the concept of a perpendicular bisector. For example, the perpendicular edge bisector of AB is the plane through the midpoint of AB, and perpendicular to it. Or equivalently, it is the set of points equidistant from A and B.



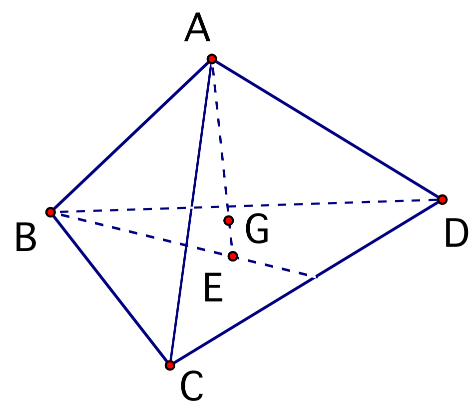
Theorem 1: The six perpendicular edge bisectors of a tetrahedron all meet at the circumcentre.
Proof: Let S be the intersection of the three perpendicular edge bisectors of AB, BC, CD. Therefore $SA = SB = SC = SD$. Therefore S must lie on all 6 perpendicular edge bisectors, and the sphere, centre S and radius SA, goes through all four vertices.

Definition 2: Let a, b, c, d denote the four faces of a tetrahedron $ABCD$. The *face angle bisector* of ab is the plane through CD bisecting the angle between the faces a, b. It is therefore also the set of points equidistant from a and b.



Theorem 2: The six face angle bisectors of a tetrahedron all meet at the incentre.

Proof: Let I be the intersection of the three face angle bisectors ab, bc, cd. Then I is equidistant from all four faces, and is therefore the centre of the insphere touching all four faces.

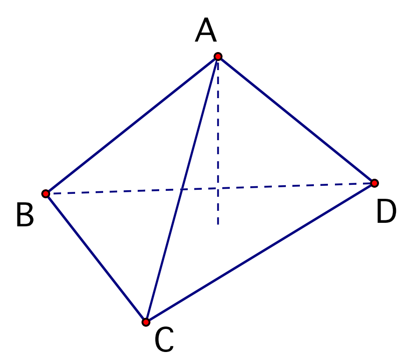


Definition 3: A median of a tetrahedron is the join of a vertex to the centroid of the opposite face.

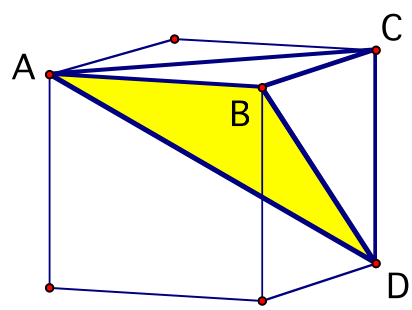
Theorem 3: The four medians of a tetrahedron meet at the centre of mass G.

Proof: Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be coordinate vectors of A, B, C, D. Then $\mathbf{e} = \frac{1}{3}(\mathbf{b} + \mathbf{c} + \mathbf{d})$ is the centroid E of BCD. Let G be the point $\mathbf{g} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$. Then G lies on the median AE because $\mathbf{g} = \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{e}$. Similarly G lies on all four medians.

To verify that G is the centre of mass of the tetrahedron, note that the line containing BE divides triangle BCD into two triangles of equal area. Therefore the plane containing ABE divides the tetrahedron into two sub-tetrahedron of equal volume (they also have the same height). Therefore the centre of mass lies in this plane, and similarly in the plane containing ACE, and hence on AE. Similarly the centre of mass lies on all the medians, and hence is G.



Definition 4: The altitude of a tetrahedron through A is the line perpendicular to face BCD.



In general the four altitudes of a tetrahedron do not meet. It suffices to give a counter-example. Consider Dehn's tetrahedron $ABCD$ inscribed in a cube as shown.

The altitudes through A, D are AB, CD which clearly do not meet.

Bibliography

Zeeman, C. (2005). *Three-dimensional theorems for schools*. Leicester: The Mathematical Association.