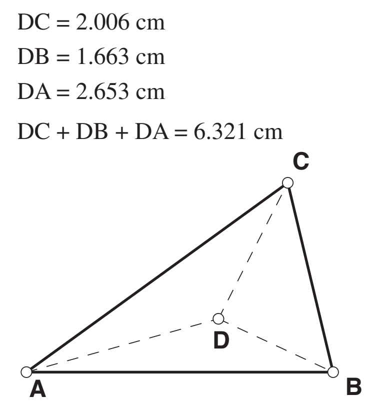
Name(s):

Airport Problem

Suppose an airport is planned to service three cities of approximately equal size. The planners decide to locate the airport so that the sum of the distances to the three cities is a minimum. Where should the airport be located?



CONJECTURE

- Open the sketch **Airport.gsp.**
- Drag point *D* until the sum of the distances to the three cities is a minimum. Search patiently and logically.
- What are the measures of angles *ADC*, *BDA*, and *CDB*?

1. What do you notice about these three angles?

- Drag *A*, *B*, or *C* to a different position, but make sure ΔABC remains acute. Again, drag *D* to obtain the optimal point for this new triangle.

2. Compare the new measurements of angles *ADC*, *BDA*, and *CDB* with those in Question 1. What do you notice?

3. Use your observations to write a conjecture.

4. **Certainty:** How certain are you that your conjecture is always true? Record your level of certainty on the number line and explain your choice.

0% 25% 50% 75% 100%

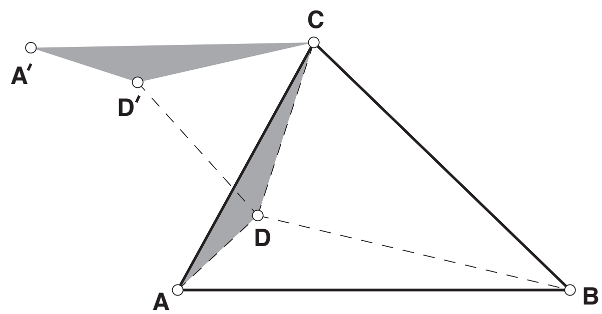
CHALLENGE If you believe your conjecture is always true, provide some examples to support your view and try to convince your partner or members of your group. Even better, support your conjecture with a logical explanation or a convincing proof. If you suspect your conjecture or your partner's conjecture is not always true, try to supply counterexamples.

PROVING

In the preceding section, you should have found that the optimal position for the airport in acute triangle *ABC* appears to be at a point connected to the vertices by lines that make angles of approximately 120°. But how certain are you?

Work through the argument below to convince yourself of your conjecture. It relies on the construction of an equivalent problem in which the optimal position is easier to locate. Follow along in your sketch if you like.

- Drag *D* to a new point inside ΔABC.



- Press the button in your sketch that rotates ΔADC by $-60°$ around point *C* to get $\Delta A'D'C$.

5. From the rotation, what can you conclude about the lengths of $\overline{CD}$ and $\overline{CD'}$?

6. What type of triangle is triangle DCD'? (*Hint:* Use the fact that angle $D'CD$ measures 60° and your conclusion in Question 5.)

7. From Question 6, what can you conclude about the lengths of $\overline{D'D}$ and $\overline{DC}$?

8. From the rotation, what can you conclude about the lengths of $\overline{AD}$ and $\overline{A'D'}$? Why?

9. What can you now conclude regarding $AD + CD + BD$ and $A'D' + D'D + DB$?

10. When will the path from A' to B $(A'D' + D'D + DB)$ be a minimum?

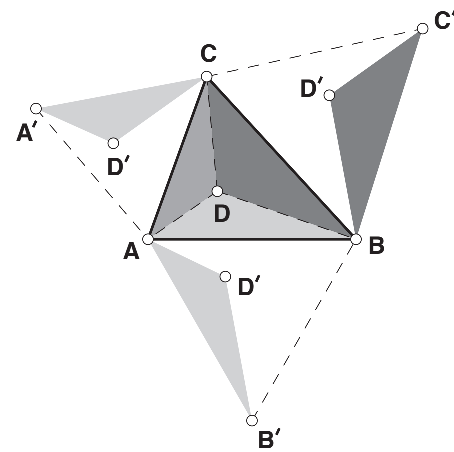
Drag D until your sketch meets the condition in Question 10. (*Hint:* It may help to construct $\overline{A'B}$.)

11. When the condition in Question 10 is met, what can you conclude about the size of angle $A'D'C$, and therefore also about angle ADC?

12. Explain how by rotating ΔCDB by $-60°$ you can prove that angle CDB measures 120°.

Looking Back

You may have noticed earlier that the optimal point for the airport is the Fermat-Torricelli point, discussed in the preceding activity. Show that the construction in this activity is equivalent to constructing equilateral triangles $A'AC$, $B'BA$, and $C'CB$ on the sides of ΔABC (see the diagram) and constructing $A'B$, $B'C$, and $C'A$ to meet at D.



Historical Notes

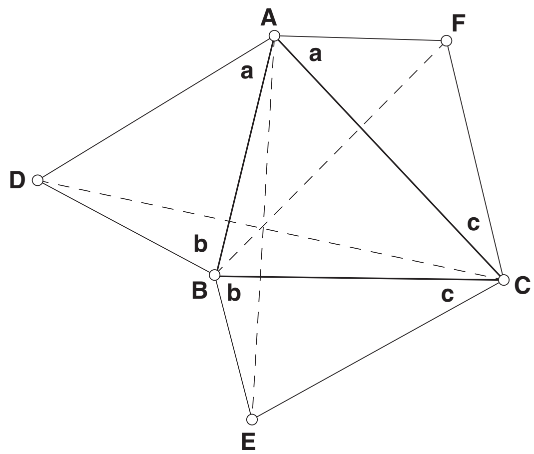
Versions of the "airport problem" and its associated geometric properties have been studied by dozens of mathematicians for the last 300 years (even though they didn't have aircraft 300 years ago!). Pierre de Fermat appears to have first posed the airport problem in an essay on optimization. He wanted to find a point inside an acute triangle such that the sum of the distances to the three vertices is a minimum. Fermat was born in 1601 and was a lawyer by profession. Although mathematics was simply an interesting hobby to him, he made important contributions to number theory, analytic geometry, calculus, and probability theory.

The Italian mathematician and scientist Evangelista Torricelli proposed constructing equilateral triangles on the sides of any triangle to locate the optimal point (see the preceding activity The Fermat-Torricelli Point). Although this solution was proposed in 1640, it was published in 1659 by Viviani, one of Torricelli's students. Torricelli is probably better known for his research into the nature of gas, which led to the invention of the mercury barometer.

The solution described in this activity was more recently invented in 1929 by the German mathematician J. Hoffman. Several other famous mathematicians—for example, C. F. Gauss and J. Steiner—have investigated the problem and have produced some interesting generalizations.

Further Exploration

1. The dynamic Sketchpad sketch of the three cities is an example of a mathematical model that can be used to represent and analyze real-world situations. However, real-world situations are extremely complex and usually have to be simplified before mathematics can be meaningfully applied to them. What are some of the assumptions that could have been made to simplify the original problem?
2. Can you relate the airport problem to the result discovered and proven in the activity Distances in an Equilateral Triangle, and use it to develop a kind of indirect proof for the optimal placement of the airport? (Use the sketch **Airport 2.gsp** to investigate the relationship.)
3. Where should the airport be placed if the cities lie in the shape of an obtuse triangle with one of the angles greater than 120°?
4. Where should the airport be placed if the three cities all lie in a straight line (are collinear)? Can you generalize?
5. Where should the airport be placed if the cities are all of different sizes, for example, if A, B, and C respectively have 60,000, 100,000, and 70,000 people?
6. Where should the airport be placed if there are four cities instead of only three? (Use the sketch **Airport 5.gsp** to investigate the problem.) Is your solution also valid if the four cities lie in the shape of a concave quadrilateral?
7. Where should a spaceport be built for four planets that lie in the shape of a tetrahedron so that the total sum of distances from the spaceport to the planets is a minimum?



"If triangles *DBA*, *ECB*, and *FAC* are constructed outwardly on the sides of any triangle *ABC* so that $m\angle DAB = m\angle CAF$, $m\angle DBA = m\angle CBE$, and $m\angle ECB = m\angle ACF$, then segments *DC*, *EA*, and *FB* are concurrent."

One proof of this generalization, given in de Villiers (1996), is a little more complicated, using Ceva's theorem. A ready-made sketch called **Fermat 3.gsp,** which illustrates this generalization, is provided. An even further generalization is given and proven in de Villiers (1999a). This involves six triangles and a sketch is given in **Fermat 4.gsp.**

AIRPORT PROBLEM (PAGE 115)

This activity can be used to reinforce the idea of proof as a means of verification, because students are likely to experience some uncertainty as to the precise location of the optimal point. This activity can be done independently of the preceding activity, The Fermat-Torricelli Point (except for the Looking Back section, which specifically refers to the Fermat-Torricelli point). Furthermore, the solution of this problem does not require knowledge of the properties of cyclic quadrilaterals—a prerequisite for The Fermat-Torricelli Point activity.

If you are planning to do The Fermat-Torricelli Point activity, it is recommended that it precede the Airport Problem. Otherwise, the logical discovery of the generalization from a right triangle to any triangle in The Fermat-Torricelli Point activity may be less surprising.

Prerequisites: Students should have some familiarity with transformations, particularly rotations.

Sketch: Airport.gsp. Additional sketches are **Airport 2.gsp, Airport 3.gsp, Airport 4.gsp,** and **Airport 5.gsp.**

CONJECTURE

1. The angle measures are all approximately equal to 120°.
2. Same as Question 1.
3. The optimal point is situated where the measures of angles *ADC*, *BDA*, and *CDB* are all equal to 120°.
4. **Certainty:** Students may have some difficulty noticing that the three angle measures are equal and may not be entirely confident that this is really always the case. This doubt thus provides a good opportunity to again reinforce the idea of proof as a means of verification.

CHALLENGE Student responses will vary.

PROVING

5. $CD = CD'$, because they map to each other.
6. Triangle DCD' is equilateral ($m\angle D'CD = 60°$ and $CD = CD'$, which imply that $m\angle CD'D$ and $m\angle CDD'$ are also both 60°).
7. $D'D = DC$.
8. $AD = A'D'$, because they map to each other.

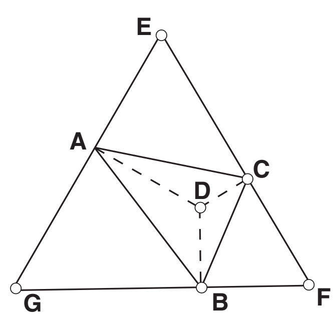
9. $AD + CD + BD = A'D' + D'D + DB$.
10. The path $A'D' + D'D + DB$ will be a minimum when it is a straight line.
11. $m\angle A'D'C = 120°$, because $A'D'D$ is a straight line and $m\angle CD'D = 60°$ (from Question 6). Therefore, $m\angle ADC = 120°$, because it maps onto angle $A'D'C$.
12. Similar to Questions 5–11.

Looking Back

$\overline{AC}$ maps to $\overline{A'C}$, so $m\angle A'CA = 60°$ and $A'C = AC$. Therefore, the other two angles of triangle $A'CA$ also measure 60°, and the triangle is therefore equilateral. Similarly, the other two triangles can be shown to be equilateral.

Further Exploration

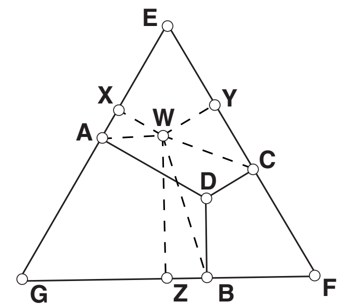
1. Some of the main assumptions are
   - The three cities are roughly the same size (if they weren't, it might be better to place the airport closer to the largest city).
   - The terrain between the cities is flat (that is, no hills or valleys).
   - There are no other natural obstacles such as rivers, swamps, or lakes to avoid.
   - The roads can be built perfectly straight.
   - The three cities lie in the shape of an acute triangle. (Actually more precisely, none of the angles is greater than 120°.)
2. **Theorem:** If triangle *EFG* is equilateral, then the angles surrounding point *D* are each 120°.



Proof: Quadrilateral *ADBG* is cyclic, since the opposite angles *GAD* and *GBD* are both right angles. But if triangle *EFG* is equilateral, then angle $AGB = 60°$ and therefore the opposite angle $ADB = 120°$. In the same way, the other two angles around *D* can be shown to be equal to 120°.

Theorem: If triangle *EFG* is equilateral, then the sum of the distances from *D* to *A, B,* and *C* is a minimum.

Proof: According to the theorem, the sum of the distances from any point *other than D* would be greater than the sum of the distances from *D* to the three cities. Let *W* be any arbitrary point not coinciding with *D*. We now want to prove that $WA + WB + WC > DA + DB + DC$.



Drop perpendiculars from *W* to the sides of the equilateral triangle. Then according to the theorem proved in the Distances in an Equilateral Triangle activity $WX + WY + WZ = DA + DB + DC$, but from the triangle inequality $WX < WA$, $WY < WC$, $WZ < WB$. Therefore, $WA + WB + WC > DA + DB + DC$.

Your students may quite rightly ask: Why another proof of the airport theorem? It might help to point out that producing a different proof often helps establish new logical connections with other results and that the purpose is therefore not of further conviction. Indeed, many theorems in mathematics have several different proofs, each providing useful links and valuable insights into why they are true.

3. The airport should be placed at the vertex of the obtuse angle.
4. The airport should be placed at the city in the middle. Similarly, if we have four collinear cities, then the airport can be placed anywhere in the middle segment. In general, for an odd number of collinear cities, the

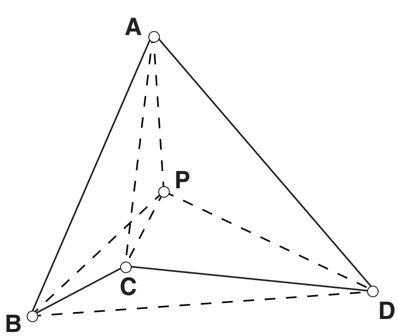
optimal solution will lie at the middle city, and for an even number of cities, the optimal solution will lie anywhere in the middle segment (see **Airport 3.gsp**).

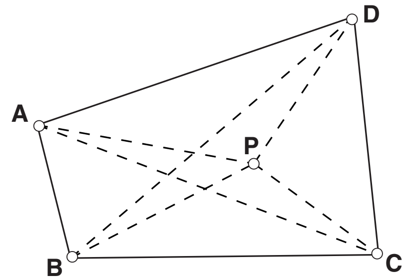
Proof: Note that the sum of distances between any two adjacent cities is always constant, irrespective of where the airport is placed between the two cities. Therefore, when the number of cities is even, the minimum sum would be found by placing the airport in the middle segment, since moving outside the middle segment will increase the sum of the distances to the middle two cities, and therefore the total sum of all the distances. Similarly, it follows that when the number of cities is odd, the optimal position is found at the middle city.

5. The easiest way to solve a problem like this is to "weigh" the distances in proportion to the sizes of the cities. For example, since the largest city is weighed the most, it ensures that the distance to the largest city will be shortened proportionally. To minimize the sum of the distances, we only need to drag D until the expression $6DA + 10DB + 7DC$ becomes a minimum (see **Airport 4.gsp**).

   However, a purely geometric solution is possible, and the author will be writing an article in this regard, which will be made available with a Sketchpad 4 sketch at his Web site http://mzone.mweb.co.za/residents/profmd/homepage.html.

6. The optimal solution for a convex quadrilateral will be at the intersection of the diagonals, but for a concave quadrilateral the optimal solution will lie at the vertex forming a *reflex* angle (angle greater than 180°).





   Consider the first figure. If P does not lie on BD, then $BP + PD > BD$ since the shortest path between two points is a straight line. Therefore to have $BP + PD$ as short as possible, P must lie somewhere on BD. Similarly, we can argue that for $AP + PC$ to be a minimum, P must also lie somewhere on AC. Therefore, point P must lie at the intersection of the two diagonals.

   Similarly, in the second case, if P does not lie on AC, then $AP + PC > AC$ since the shortest path between two points is a straight line. Therefore to have $AP + PC$ as short as possible, P must lie somewhere on AC. But since BD in this case lies outside the quadrilateral, P should be placed on AC as close as possible to BD, therefore at vertex C.

7. If the space coordinates of the four vertices A, B, C and D are respectively (x_B, y_B, z_B), (x_C, y_C, z_C), and (x_D, y_D, z_D) and that of the airport is (x, y, z), then we just need to minimize the expression

$$\sqrt{(x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2} + \sqrt{(x - x_B)^2 + (y - y_B)^2 + (z - z_B)^2} + \sqrt{(x - x_C)^2 + (y - y_C)^2 + (z - z_C)^2}$$

   Although this would require advanced calculus, we could, if we had a three-dimensional version of Sketchpad, simply drag the point representing the airport until a minimum is obtained.