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96.25 Going halfway with circular boundaries

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$A^*(-\Sigma+\alpha, \Sigma+\alpha, \Sigma+\alpha)$, $B^*(\Sigma+\beta, -\Sigma+\beta, \Sigma+\beta)$, $C^*(\Sigma+\gamma, \Sigma+\gamma, -\Sigma+\gamma)$ follow easily, leading to the meet of $A'A^*$, $B'B^*$, $C'C^*$ at the point $E(\Sigma + \Pi - 2\Sigma\beta\gamma, \Sigma + \Pi - 2\Sigma\gamma\alpha, \Sigma + \Pi - 2\Sigma\alpha\beta)$ by determinants.

Since the orthocentre has coordinates $H[\beta\gamma, \gamma\alpha, \alpha\beta]$, the Exeter point E clearly lies on the Euler line GH of the host triangle ABC. We next consider the Euler line of the tangential triangle $A'B'C'$.

Now although the centroid

$$G'(\Sigma + 2\Pi - \alpha - 2\Sigma\beta\gamma, \Sigma + 2\Pi - \beta - 2\Sigma\gamma\alpha, \Sigma + 2\Pi - \gamma - 2\Sigma\alpha\beta)$$

is easily found, the location of the circumcentre

$$O'(\Sigma + 3\Pi - (3\Sigma + \Pi)\beta\gamma, \Sigma + 3\Pi - (3\Sigma + \Pi)\gamma\alpha, \Sigma + 3\Pi - (3\Sigma + \Pi)\alpha\beta),$$

which again lies on the Euler line $G'H'$, is more difficult. One approach is to translate the collinearity of the incentre, Gergonne point and de Longchamps point ([3]) to the tangential triangle and then to proceed along the Euler line $G'H'$ to O'. Thus the coordinates of the nine-points centre N' may be deduced from G' and O' and are

$$N'(3\Sigma + 5\Pi - 4\alpha - (5\Sigma - \Pi)\beta\gamma, 3\Sigma + 5\Pi - 4\beta - (5\Sigma - \Pi)\gamma\alpha, 3\Sigma + 5\Pi - 4\gamma - (5\Sigma - \Pi)\alpha\beta).$$

Next observe that the circumcircle of the triangle of reference ABC is the incircle of the tangential triangle $A'B'C'$.

Then another determinant shows that the point

$$F'(\Sigma + 3\Pi - 2\alpha - 2\Sigma\beta\gamma, \Sigma + 3\Pi - 2\beta - 2\Sigma\gamma\alpha, \Sigma + 3\Pi - 2\gamma - 2\Sigma\alpha\beta)$$

lies on ON', and we see further (since these coordinates satisfy $\# = 0$ and $N' \notin F'O$) that F' is indeed the Feuerbach point for the tangential triangle.

Finally, the collinearity $EG'F'$ is easily verified.

References

1. G. C. Smith and G. Leversha, Euler and triangle geometry, *Math. Gaz.* **91** (November 2007) p. 436.
2. C. J. Bradley, *Challenges in geometry*, Oxford (2005) appendix A.
3. J. A. Scott, Miscellaneous triangle properties, *Math. Gaz.* **94** (March 2010) p. 103.

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96.27 Proving a nineteenth century ellipse identity

Duncan F. Gregory (1813-1844), a great-great-grandson of James Gregory of refracting telescope and infinite series fame, was a founding editor of the *Cambridge Mathematical Journal*. After his friend and colleague Samuel S. Greatheed left Cambridge, Gregory maintained a mathematical, as well as a personal, correspondence with him. Unfortunately, we have only six letters, all from Gregory to Greatheed; none of Greatheed's replies survives. In several of Gregory's letters, which are in the Wren Library at Trinity College, Cambridge, Gregory posed mathematical problems. In an undated letter, written sometime between February and July 1839, Gregory posed an intriguing geometric problem and included a diagram, [1].

To try you – here is a mathematical nut for you to crack –

‘ F, f are the foci of an ellipse to which AB, BC, AC are tangents $FA, B, C - fA', B', C'$ (are drawn). Then

$$\frac{x}{b}\cdot\frac{x'}{c} + \frac{y}{a}\cdot\frac{y'}{c} + \frac{z}{b}\cdot\frac{z'}{a} = 1.\text{'}$$

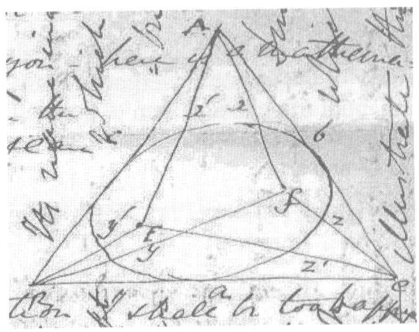

FIGURE 1

We don't know if either Gregory or Greatheed ‘cracked the nut’, but there is no proof of the problem to be found in the *Cambridge Mathematical Journal*, nor were we successful in a search for a proof in modern sources.

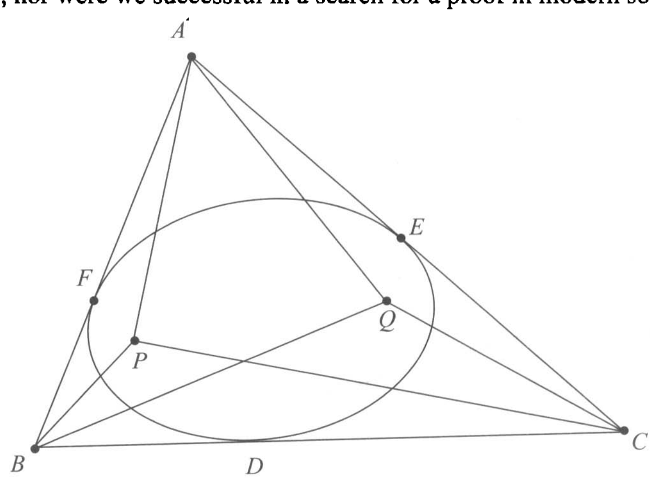

FIGURE 2

We have changed the lettering for the purposes of this paper and re-created the diagram. The problem now reads as follows.

‘ Let P, Q be the foci of an ellipse, which is inscribed in triangle ABC so that the sides of the triangle are tangent to the ellipse (Figure 2). Then the following identity holds:

$$\frac{PA \cdot QA}{CA \cdot AB} + \frac{PB \cdot QB}{AB \cdot BC} + \frac{PC \cdot QC}{BC \cdot CA} = 1.\text{'} \qquad (1)$$

The proof that follows, which is geometric and employs the physical properties of the ellipse, is very likely the type of proof that would have been seen at the time.

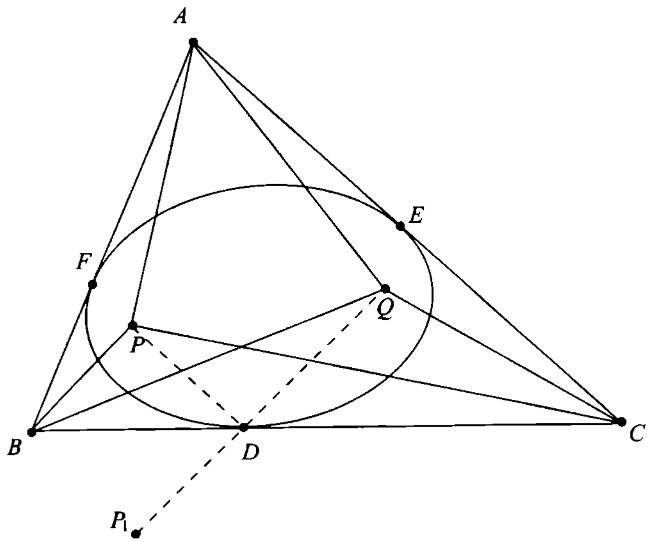

FIGURE 3

Proof: The ellipse has three tangents BC, CA and AB, with corresponding tangent points D, E and F. We draw PD and QD and extend QD to point P_1 to make $DP_1 = DP$ (Figure 3).

Since BC is tangent to the ellipse at point D, by the optical property of ellipses we have $\angle PDB = \angle P_1DB$, and because $DP_1 = DP$, P_1 is the symmetric point of P about the line BC. Thus the length of the major axis of the ellipse = $QD + DP = QP_1$.

In a like manner, we draw PE, QE, PF and QF and extend QE and QF to P_2 and P_3 respectively to make $EP_2 = EP$, $FP_3 = FP$. Similarly, P_2 and P_3 are symmetric points of P with respect to the lines CA and AB, and the length of the major axis of the ellipse = $QP_1 = QP_2 = QP_3$, and thus Q is the circumcentre of the triangle $P_1P_2P_3$. We then connect $A\,P_3\,B\,P_1\,C\,P_2\,A$ (Figure 4).

Because P_2 and P_3 are the symmetric points of P with respect to CA and AB, $AP_2 = AP = AP_3$, and also $QP_2 = QP_3$. Therefore we have a pair of congruent triangles, AP_2Q and AP_3Q, which are, of course, equal in area, i.e. $S_{\Delta AP_2Q} = S_{\Delta AP_3Q}$.

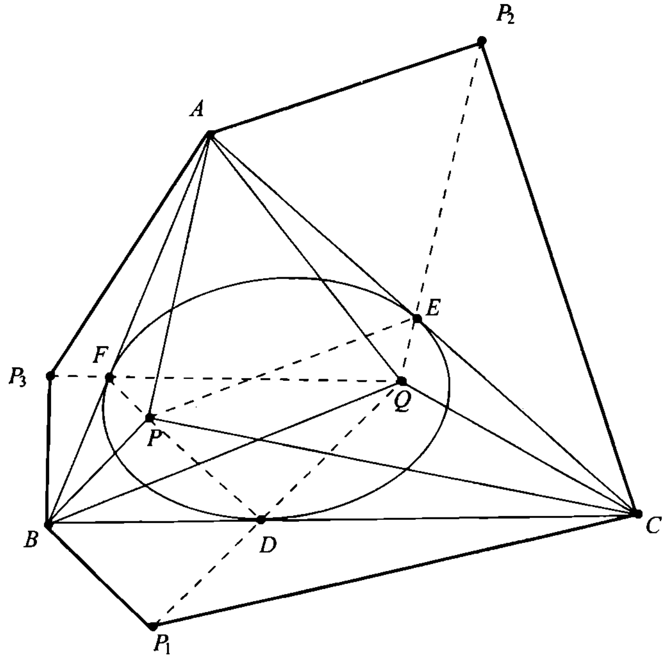

FIGURE 4

We then show $S_{\Delta AP_2Q} = S_{\Delta AP_3Q} = \frac{1}{2}PA.QA \sin \angle BAC$.

In fact, we have

$$\angle P_2AC = \angle PAC \Rightarrow \angle P_2AC = \angle PAQ + \angle CAQ \tag{1}$$

$$\angle P_3AB = \angle BAP \tag{2}$$

$$\angle P_2AQ = \angle P_3AQ \Rightarrow \angle P_2AC + \angle CAQ = \angle P_3AB + \angle BAP + \angle PAQ \tag{3}$$

Substituting (1) in (3) we get

$$\angle PAQ + 2\angle CAQ = \angle P_3AB + \angle BAP + \angle PAQ. \tag{4}$$

Substituting (2) in (4) we get

$$\angle CAQ = \angle BAP = \angle P_3AB.$$

Thus $\angle P_3AQ = \angle BAC$, and we obtain

$$S_{\Delta AP_2Q} = S_{\Delta AP_3Q} = \tfrac{1}{2}P_3A.QA \sin \angle P_3AQ = \tfrac{1}{2}PA.QA \sin \angle BAC$$

and

$$\frac{PA.QA}{CA.AB} = \frac{\frac{1}{2}PA.QA \sin \angle BAC}{\frac{1}{2}CA.AB \sin \angle BAC} = \frac{S_{\Delta AP_2Q}}{S_{\Delta ABC}} = \frac{S_{\Delta AP_3Q}}{S_{\Delta ABC}} = \frac{S_{\Delta P_2QP_3}}{S_{\Delta ABC}}. \tag{5}$$

In a similar manner, we have

$$S_{\Delta BP_3Q} = S_{\Delta BP_1Q} = \tfrac{1}{2}PB.QB \sin \angle CBA, S_{\Delta CP_1Q} = S_{\Delta CP_2Q} = \tfrac{1}{2}PC.QC \sin \angle ACB,$$

and

$$\frac{PB.QB}{AB.BC} = \frac{S_{BP_3QP_1}}{2S_{\Delta ABC}}, \tag{6}$$

$$\frac{PC.QC}{BC.CA} = \frac{S_{CP_1QP_2}}{2S_{\Delta ABC}}. \tag{7}$$

By combining (5),(6),(7) and adding them, we obtain

$$\frac{PA.QA}{CA.AB} + \frac{PB.QB}{AB.BC} + \frac{PC.QC}{BC.CA} = \frac{S_{AP_3BP_1CP_2}}{2S_{\Delta ABC}}.$$

Note that since P_1 is the symmetric point of P with respect to line BC, we have $S_{\Delta CP_1B} = S_{\Delta BPC}$; for the same reasons $S_{\Delta CP_2A} = S_{\Delta CPA}$, $S_{\Delta AP_3B} = S_{\Delta APB}$, and therefore

$$\frac{PA.QA}{CA.AB} + \frac{PQ.QB}{AB.BC} + \frac{PC.QC}{BC.CA} = \frac{S_{AP_3BP_1CP_2A}}{2S_{\Delta ABC}} = \frac{2S_{\Delta ABC}}{2S_{\Delta ABC}} = 1.$$

Since the circle is a special case of the ellipse, we find that this theorem holds for circles as well.

Corollary: Setting O as the centre of a circle inscribed in the triangle ABC (Figure 5), then the identity

$$\frac{OA^2}{CA.AB} + \frac{OB^2}{AB.BC} + \frac{OC^2}{BC.CA} = 1$$

holds.

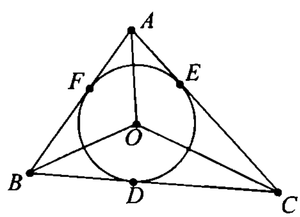

FIGURE 5

Reference

1. Gregory 1839 (*Add. Ms. Letter* 1/137), used with permission of the Master and Fellows of Trinity College, Cambridge.

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