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NOTE

An interesting theorem related to a hexagon with opposite sides that are parallel

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ABSTRACT
It's often useful extending students beyond the limiting geometry of triangles and quadrilaterals to regularly consider generalizations of results for triangles and quadrilaterals to higher order polygons. A brief heuristic description is given here of the author applying this strategy, and which led to an interesting result related to the perpendicular bisectors of a hexagon with opposite sides parallel, and a rather novel proof. It provides a nice challenge and exploration for students to investigate using dynamic geometry.

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1. Introduction

One of the disadvantages of high school (and some college) geometry curricula in many countries is the limitation to only investigating triangles and quadrilaterals. This often creates various geometric misconceptions in students' minds. For example, I've frequently found university students as well as practising mathematics teachers often exhibiting misconceptions like the following, to name but a few:

(1) if a polygon has opposite sides parallel, then it's opposite sides are equal (incorrectly generalizing from a parallelogram)
(2) if a polygon has opposite sides equal, then it's opposite sides are parallel (incorrectly generalizing from a parallelogram)
(3) if a polygon has all sides equal, then all its angles are equal (incorrectly generalizing from a triangle and not even thinking of a rhombus, which already refutes the statement)
(4) if a polygon has all angles equal, then all its sides are equal (incorrectly generalizing from a triangle and not even thinking of a rectangle, which already refutes the statement)

Often the responses of students and teachers to the above are motivated by visual stereotypes of regular polygons, and they apparently find it hard to imagine say, a hexagon with opposites sides parallel, but not necessarily equal. Similarly, they appear to struggle with

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conceptualizing a concave pentagon or a pentagon that has some sides parallel to each other.

However, generalization is a valuable process in mathematics as it increases our understanding of a topic by identifying its essential features and looking at it from a higher vantage point. So for example, looking at the solution of algebraic equations from the perspective of abstract algebra of groups and rings shows us why quintic polynomials do not have algebraic solutions, and a comparison of a simple concept such as 'straight line' in the Euclidean plane with the equivalent concept on other surfaces, deepens our understanding of 'straightness'. Two- and three-dimensional space together with the theorem of Pythagoras in its definition of distance as a metric, take on different meanings when we generalize to multi-dimensional space. The same is true if we generalize triangles and quadrilaterals to other polygons, and this practice should be encouraged not only at school, but also in mathematics teacher training programmes.

It deepens understanding of what 'cyclic' or 'circumscribed' means when generalizing from results such as 'a quadrilateral is cyclic, if and only if, its perpendicular bisectors are concurrent' and 'a quadrilateral is circumscribed, if and only if, its angle bisectors are concurrent' respectively to 'cyclic polygons' and 'circumscribed polygons'. An advantage of pushing students to continually generalize is that it provides not only a natural platform for interesting investigations, but also of proof and disproof. For example, as shown in (de Villiers, 2016), though the result that 'the *alternate* angles of a (convex) cyclic quadrilateral are supplementary' easily generalizes to 'any (convex) cyclic $2n$-gon has the sum of its *alternate* angles equal to $(n-1)180°$', the converse is not true and can be easily disproved with a counter-example (except for a quadrilateral).

2. Investigating perpendicular bisectors of polygons with opposite sides parallel

Recently I was applying this strategy of generalizing myself as follows. If one constructs the perpendicular bisectors of the sides of a parallelogram *ABCD* one will find that it forms another parallelogram *EFGH* as shown in Figure 1. This follows directly from the point symmetry of the parallelogram and the symmetry of the construction.[1]

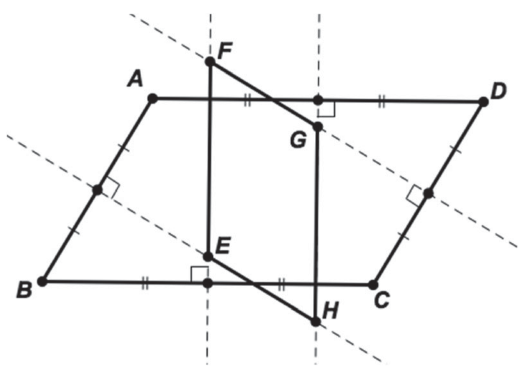

Figure 1. Perpendicular bisectors of parallelogram.

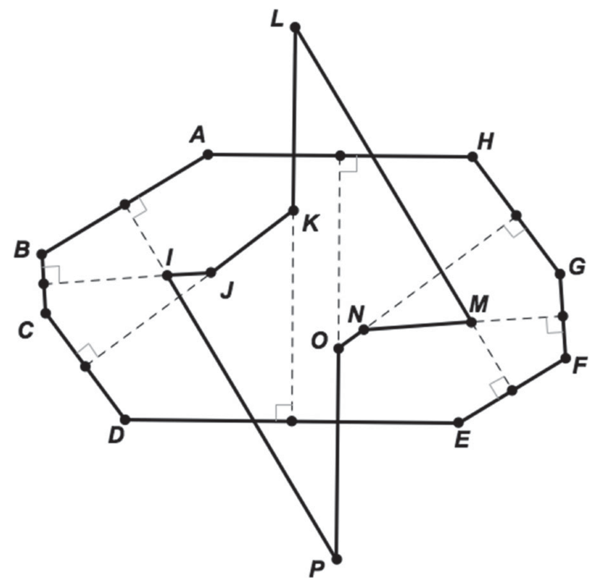

Figure 2. Perpendicular bisectors of an octagon with opposite sides parallel

A generalization to polygons with opposite sides parallel is immediately apparent as follows: 'The perpendicular bisectors of the sides of a polygon with opposite sides parallel forms another polygon with opposite sides parallel.' An illustrative example for an octagon with opposite sides parallel is given in Figure 2.

A general proof follows from the simple observation that pairs of perpendiculars to pairs of opposite parallel sides are also parallel to each other. Also note that an equivalent formulation of this generalization is the following: 'The respective circumcentres of triangles $A_1A_2A_3$; $A_2A_3A_4$; ... $A_nA_1A_2$ of a polygon $A_1A_2A_3 \dots A_n$ with opposite sides parallel, forms another polygon with opposite sides parallel.'

Much to my surprise, however, I found that when I constructed a dynamic geometry sketch for a hexagon *ABCDEF* with opposite sides parallel as shown in Figure 3, then the hexagon *GHIJKL* formed by its perpendicular bisectors not only had opposite sides parallel as expected, but also equal as shown by the displayed measurements of the side lengths. In other words, a *parallelo-hexagon* is formed. But *why* was this true?

Before continuing, readers are now invited to view and investigate a dynamic version of this sketch online by selecting and dragging any of the points *A*, *B*, *C*, *E* or *F*. Go to: http://dynamicmathematicslearning.com/perp-bisectors-parallel-hexagon.html

3. Explaining (proving) the conjecture

Having done a dynamic geometry sketch and explored it with dragging, I had no doubt that the conjecture was true. However, I desired an explanation, a deeper understanding of why the conjecture was true, which no amount of dragging could satisfy – all that showed was that it was true. So for me the main purpose of a proof in this instance was not that of seeking verification, but that of illumination and deeper insight (de Villiers, 1990).

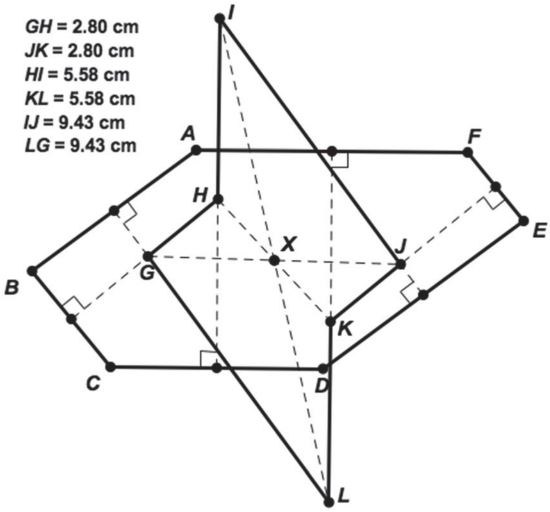

Figure 3. Perpendicular bisectors of a hexagon with opposite sides parallel

Though it's a result that probably lends itself directly to attack by vectors or by complex algebra, I instead wanted to prove it using only geometry as I personally felt that would be more explanatory. More-over, even though one can easily verify the truth of the conjecture with advanced symbolic software such as *Mathematica* (see example in Appendix), such a computer proof provides no insight into why it is true; only confirms that which I've already observed through extensive exploration using dynamic geometry software.

At the beginning I was quite optimistic that I would find an easy straightforward proof, given that several parallelograms are formed and anticipated that it would be relatively easy to prove that the main diagonals *GJ*, *HK*, and *IL* of *GHIJKL*, were concurrent at *X*, which would establish that it was the centre of symmetry of the formed hexagon, and that it was therefore a parallelo-hexagon. But proving this was not as easy as I'd thought. I then switched to trying to use a concurrency theorem related to homothetic polygons, as well as the theorem of Desargues, but still failed to show that these various points of concurrency coincided.

At long last, I switched to a Polya strategy I've always taught to my students, namely, to try and make some construction that would relate the problem to a result already known or easily proved (Polya, 1945).

From the point (and half turn) symmetry of a parallelo-hexagon, it immediately follows that the perpendicular bisectors of its sides would form another parallelo-hexagon. Was there a way of doing some construction on the hexagon *ABCDEF* in Figure 3, to convert it to a parallelo-hexagon having the same perpendicular bisectors? Achieving that would immediately prove (and explain) the result!

After some thought I came up with the construction shown in Figure 4 that achieves this. Choosing any pair of opposite sides, construct a point *X* on the perpendicular bisector of one of the sides – in Figure 4, the side *CD* has been chosen. Then construct a circle with *X* as centre and *XC* as radius. Choose a point *Y* on the perpendicular bisector of *CD* and draw a line through *Y* parallel to *CD* to intersect the circle in C_1 and D_1 to ensure that

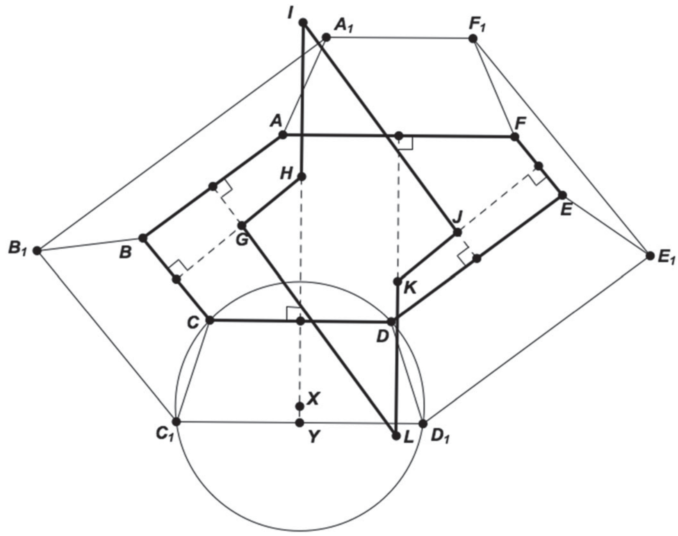

Figure 4. Auxiliary construction and proof

$C_1D_1 > CD$. In other words, choose Y so that $d(X, C) < d(X, CD)$. Therefore, CC_1D_1D is an isosceles trapezium and its axis of symmetry coincides with the perpendicular bisector of CD.

Next reflect the point D_1around the perpendicular bisector of sides DE, and then continue reflecting each subsequent reflection in the respective perpendicular bisectors of EF, FA and AB. By connecting all these reflected points in sequence we have now constructed isosceles trapezia on sides CD, DE, EF, FA and AB, and now have to prove that BB_1C_1C is also an isosceles trapezium.

From the repeated reflections, it follows that $BB_1 = CC_1$. To prove that BB_1C_1C is also an isosceles trapezium, we shall now show that the angles B_1BC and C_1CB are also equal. Let angles FAB, ABC and BCD respectively be equal to x, y and z. Since a hexagon with opposite sides parallel has opposite angles equal, it follows that the opposite angles CDE, DEF and EFA are respectively also x, y and z. Let $\angle\ C_1CD = p = \angle\ D_1DC$, then $\angle\ C_1CB = 360 - z - p$ using the sum of angles around a point. From the reflections, it further follows that $\angle\ D_1DE = \angle\ E_1ED = 360 - x - p$, $\angle\ E_1EF = \angle\ F_1FE = x + p - y$, $\angle\ F_1FA = \angle\ A_1AF = 2y - p$ (also using $x + y + z = 360$ if $ABCDEF$ is convex), $\angle\ A_1AB = \angle B_1BA => \angle B_1BC = 360 - z - p$. Hence, $\angle B_1BC = \angle C_1CB$ from which follows that BB_1C_1C is also an isosceles trapezium. In other words, we have now constructed a hexagon $A_1B_1C_1D_1E_1F_1$ with its corresponding sides parallel to $ABCDEF$.

Now note that in Figure 4, $C_1D_1 > CD$ and $F_1A_1 < FA$. As Y is moved toward the midpoint of CD, C_1D_1 becomes smaller until it becomes equal to CD, while F_1A_1 increases to become equal to FA. Note that by moving X further and further away from CD and thus enlarging the circle, we can let C_1D_1 be any given value greater than CD (and likewise F_1A_1 any given value smaller than FA).[2] Since $FA > CD$ in the diagram in Figure 4, it follows from the intermediate value theorem that at some position of Y, $C_1D_1 = F_1A_1$. However, if a hexagon with opposites sides parallel has one pair of opposite sides equal, then the other two pairs of sides are also equal.[3] Hence, $A_1B_1C_1D_1E_1F_1$ will have become a

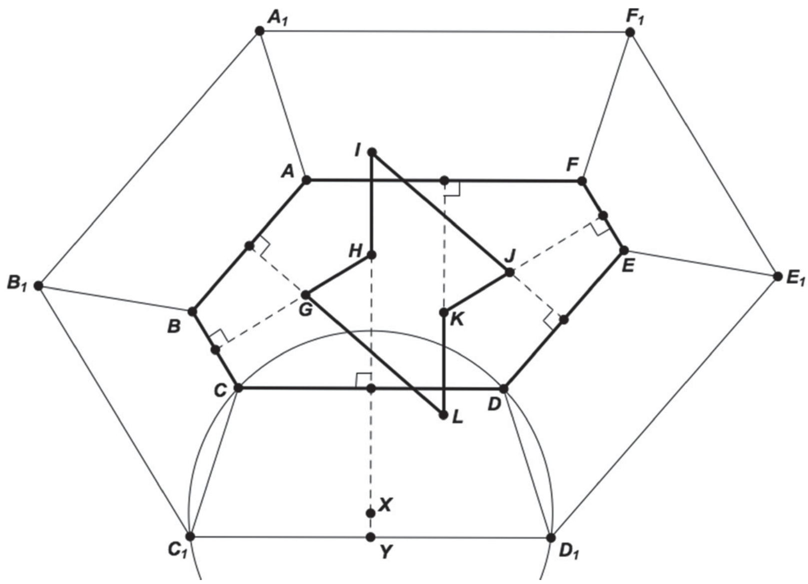

Figure 5. Exterior constructed hexagon with all sides greater than original

parallelo-hexagon; i.e. a hexagon with opposites side parallel and equal. From its point (and half-turn) symmetry we know that its perpendicular bisectors will form a hexagon *GHIJKL* with opposite sides parallel and equal. But by construction, the perpendicular bisectors of $A_1B_1C_1D_1E_1F_1$ coincide with that of the original hexagon *ABCDEF*, and therefore completes the proof.

Finally, my conjecture has now become the following theorem: 'The perpendicular bisectors of a hexagon with opposite sides parallel produce a hexagon with opposite sides equal and parallel'.

The given proof is rather novel in that it not only uses an auxiliary construction, but also the intermediate value theorem of calculus.

4. Weaknesses in the argument

A first weakness of the argument above is obviously its visual dependency on the relations displayed in the specific diagram in Figure 4. Since $\angle F_1FA = \angle A_1AF = 2y - p$, if $y > 90°$, we can have the situation shown in Figure 5 where $F_1A_1 > FA$. However, by moving X further and further away from CD as shown in Figure 6 to enlarge the circle, and thus increasing the value of y (while also changing C_1D_1 by appropriately moving Y), we can still find a position where $F_1A_1 < FA$, and the same argument as before will apply.

The argument is also dependent on the 'dynamic movement' of points X and Y, and though the construction still holds, it would also need to be adapted and modified a bit for the tricky cases when *ABCDEF* becomes concave or crossed (for which the result remains true as shown in Figure 7).

For that reason, even though a proof by vectors or complex algebra may not be as 'explanatory' (for me personally) as the one above, such proofs are definitely superior in

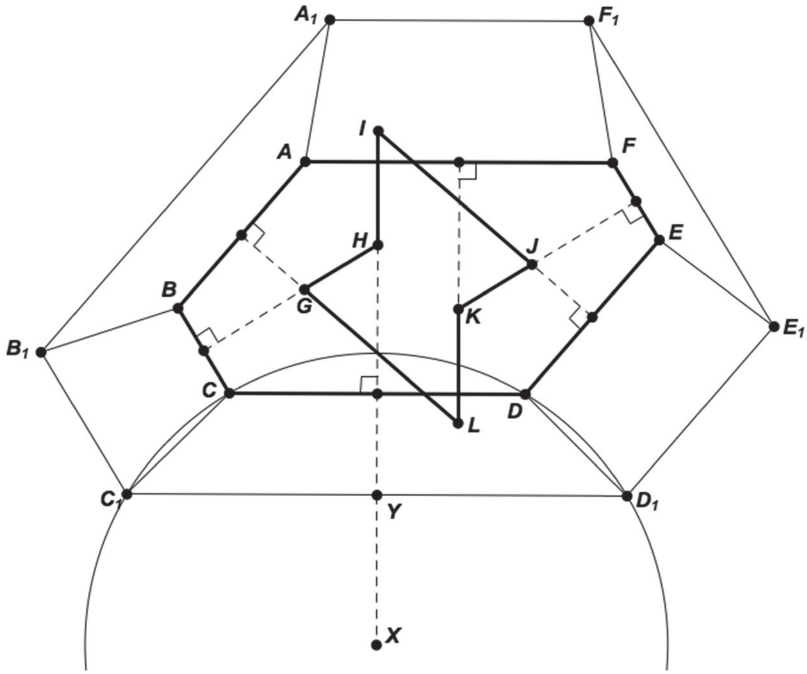

Figure 6. Exterior constructed hexagon with one side smaller than original

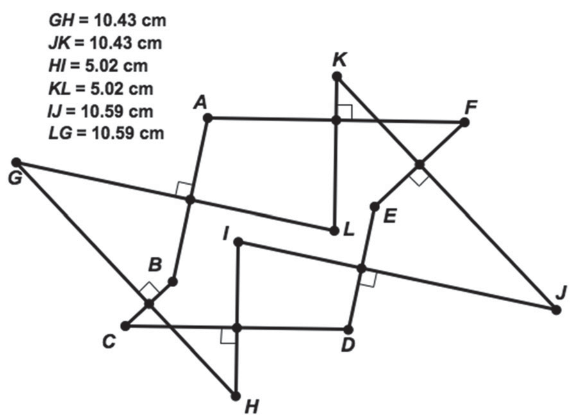

Figure 7. Concave hexagon *ABCDEF*

terms of automatically establishing the result in all possible cases. The same applies to a computer proof using software such as *Mathematica*.

5. Concluding comments

It is not unlikely that others might easily find purely geometric proofs without having to rely on the rather elaborate argument and proof produced here. However, firstly providing such an investigation for students, as well as this proof for the hexagon case, has some educational benefit in showing the value of trying to relate a result that one wants to prove to something that one already knows. The strategy is similar to the one used for proving that the segment connecting the midpoints of two sides of a triangle are parallel to and is half the length of the third side. In that case, one relates the result by an auxiliary construction to well-known results of a parallelogram.

Disclosure statement

No potential conflict of interest was reported by the authors.

Notes

1. Of additional interest is that the formed parallelogram *EFGH* in Figure 1 is similar to *ABCD*, but the proof is left as an exercise to the reader.
2. If need be, we can also make C_1D_1 be any given value smaller than *CD* by moving *Y* to the other side of *CD* into the smaller circle sector (and likewise F_1A_1 any given value larger than *FA*).
3. The proof of this result is straightforward and left to the reader.

References

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Polya, G. (1945). *How to solve it*. Princeton University Press.

APPENDIX: *Mathematica* Proof of Conjecture

The *Mathematica* proof below was kindly done for me a colleague, Dirk Basson, in the Department of Mathematics at the University of Stellenbosch.

Here is the **computer code**:

```
ClearAll['Global'*'];
    d1 = (1-c1)/(1-c3);
    d2 = (c1-c2)/(1-c3);
    d3 = (c2-c3)/(1-c3);
    e1 = d1 + c1;
    e2 = d2-c1;
    e3 = d3;
    f1 = e1;
    f2 = e2 + c2;
    f3 = e3-c2;
    a = −1/d1;
    b = d2 a;
    FF = f1AA + f2BB + f3CC;
    EE = e1AA + e2 BB + e3CC;
    DD = d1AA + d2BB + d3CC;
    Ap = (AA + BB);
    Bp = (BB + CC);
    Cp = (CC + DD);
    Dp = (DD + EE);
    Ep = (EE + FF);
    Fp = (FF + AA);
    Gdot = Ap BB – Ap AA + Bp CC-Bp BB;
```

```
Jdot = Dp BB – Dp AA+ Ep CC – Ep BB;
Hdot = a Cp DD – a Cp CC + b Bp CC – b Bp BB;
Kdot = a Fp DD – a Fp CC + b Ep CC – b Ep BB;
Simplify[Expand[Gdot + Jdot – Hdot – Kdot]];
f[AA_,BB_,CC_] = Gdot + Jdot – Hdot – Kdot;
Simplify[f[0,0,1](1-c1)]
Simplify[f[0,1,0](1-c1)]
Simplify[f[1,0,0](1-c1)]
Simplify[f[1,1,0](1-c1)]
Simplify[f[1,0,1](1-c1)]
Simplify[f[0,1,1](1-c1)]
```

The **received output** is 0, 0, 0, 0, 0, 0. The first three zeros mean that that the coefficients of A.A, B.B and C.C are zero, and the last three therefore imply that the coefficients of A.B, B.C and C.A are also zero. This shows that the formed hexagon has opposite sides parallel and equal.