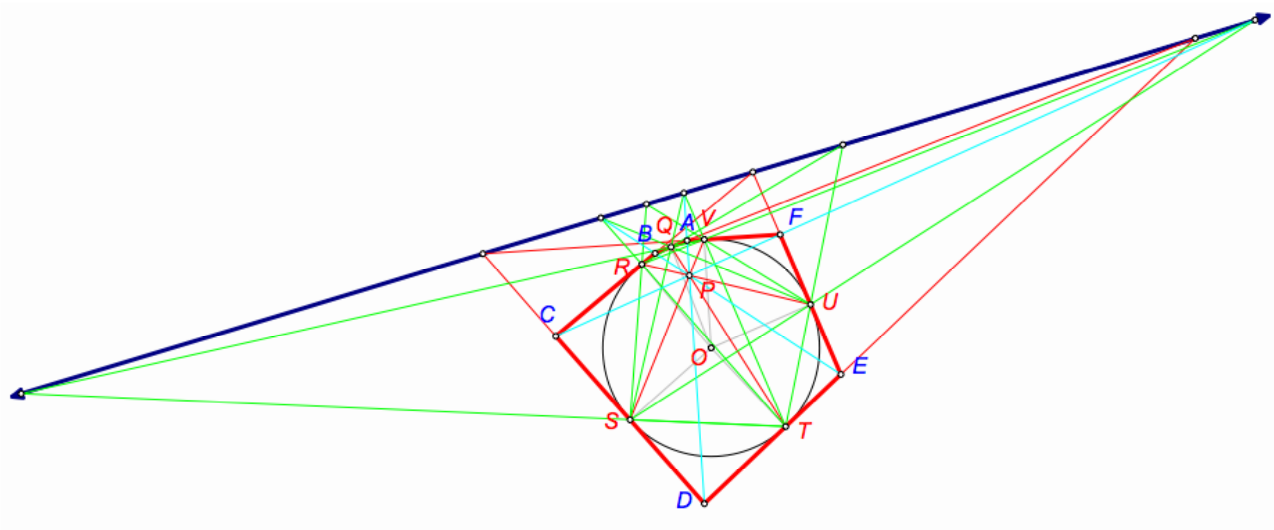
BMO Concurrency problem



The hexagon *ABCDEF* circumscribes a circle whose centre is *O*. The circle touches *AB*, *CD*, *EF* at their midpoints *Q*, *S*, *U*, and touches *BC*, *DE*, *FA* at *R*, *T*, *V*. We prove that *QT*, *SV*, *UR* are concurrent in a point *P*, say, and that *AD*, *BE*, *CF* also pass through *P*.

Proof: The two tangents from a point to a circle are equal, so *AV* = *AQ* = *BQ* = *BR*. Thus angles *VOQ* and *QOR* are equal. Then by the property of angles at the centre we have angles *VTQ* and *QTR* are equal, thus *QT* bisects angle *VTR* in triangle *VTR*. Similarly *SV* bisects angle *RVT*, and *UR* bisects angle *TRV*. But the bisectors of the angles of a triangle are concurrent. Hence *QT*, *SV*, *UR* meet in a point *P*, say. Note that *P* is the incentre of *VTR*.

It follows that triangles *VQR*, *STU* are perspective (with centre *P*), hence by Desargues' perspective theorem (*QR*.*TU*), (*RV*.*US*), (*VQ*.*ST*) are collinear points, lying on a line *l*, say.

But hexagon *QRSTUV* is inscribed in a conic, so by Pascal's theorem the intersections (*QR*.*TU*), (*RS*.*UV*), (*ST*.*VQ*) are concurrent. Since two of them lie on *l*, so does (*RS*.*UV*).

Hexagon *QRVTUS* gives the collinear points (*QR*.*TU*), (*RV*.*US*), (*VT*.*SQ*). Again two of these are known to lie on *l*, hence (*VT*.*SQ*) is also on *l*.

Finally *QUVTRS* gives (*QU*.*TR*), (*UV*.*RS*), (*VT*.*SQ*) collinear, whence (*QU*.*TR*) is on *l*. We therefore have the six intersections (*QR*.*TU*), (*QS*.*TV*), (*QU*.*RT*), (*QV*.*ST*), (*RS*.*UV*), (*RV*.*SU*) lying on *l*.

Now consider the degenerate hexagon *QQRTTU*, whose six sides are the tangent at *Q* (i.e. the line *AB*), *QR*, *RT*, the tangent at *T* (i.e. *DE*), *TU*, *UQ*. Pascal's theorem shows that (*AB*.*DE*), (*QR*.*TU*), (*RT*.*UQ*) are collinear. But these last two points lie on *l*, hence (*AB*.*DE*) is on *l*. Similarly, (*BC*.*EF*) and (*CD*.*FA*) lie on *l*. It follows that *l* is the polar of *P* with respect to the circle.

In triangles *QAV*, *TDS* we have (*QA*.*TD*) = (*AB*.*DE*) and lies on *l*; (*AV*.*DS*) = (*FA*.*CD*), and is on *l*. We have already shown that (*VQ*.*ST*) lies on *l*, thus *l* is the axis of perspective, and *QAV*, *TDS* are perspective. It follows that *QT*, *AD*, *VS* are concurrent. But *QT*, *VS* intersect in *P*, hence *AD* passes through *P*. Similarly *BE*, *CF* pass through *P*.

Since (*AB*.*DE*), (*BC*.*EF*), (*CD*.*FA*) are collinear, the points *A*, *B*, *C*, *D*, *E*, *F* lie on a conic. Thus *ABCDEF* circumscribes the circle and is inscribed in another conic.

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