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CENTROIDS AND SOME CHARACTERIZATIONS OF PARALLELOGRAMS

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Abstract. For a polygon P, we consider the centroid G_0 of the vertices of P, the centroid G_1 of the edges of P and the centroid G_2 of the interior of P, respectively. When P is a triangle, the centroid G_0 always coincides with the centroid G_2. For the centroid G_1 of a triangle, it was proved that the centroid G_1 of a triangle coincides with the centroid G_2 of the triangle if and only if the triangle is equilateral.

In this paper, we study the relationships between the centroids G_0, G_1 and G_2 of a quadrangle P. As a result, we show that parallelograms are the only quadrangles which satisfy either $G_0 = G_1$ or $G_0 = G_2$. Furthermore, we establish a characterization theorem for convex quadrangles satisfying $G_1 = G_2$, and give some examples (convex or concave) which are not parallelograms but satisfy $G_1 = G_2$.

1. Introduction

Let P denote a polygon in the plane $\mathbb{R}^2$. Then we consider the centroid (or center of mass, or center of gravity, or barycenter) G_2 of the interior of P, the centroid G_1 of the edges of P and the centroid G_0 of the vertices of P. The centroid G_1 of the edges of P is also called the perimeter centroid of P ([2]).

If P is a triangle, then the centroid G_1 coincides with the center of the Spieker circle, which is the incircle of the triangle formed by connecting midpoint of each side of the original triangle P ([1, p. 249]).

For a triangle P, we have the following ([11, Theorem 2]).

Proposition 1.1. *Let ABC denote a triangle. Then we have*:

(1) $G_0 = G_2(= G)$, *where* $G = (A + B + C)/3$.
(2) $G_1 = G_2$ *if and only if the triangle ABC is equilateral.*

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Hence, it is quite natural to ask the following:

Which quadrangles satisfy one of the conditions $G_0 = G_1$, $G_0 = G_2$ and $G_1 = G_2$?

In this paper, first of all, in Section 3 we answer the above question as follows.

Theorem A. *Let P denote a quadrangle. Then the following are equivalent.*

(1) *P satisfies $G_0 = G_1$.*
(2) *P satisfies $G_0 = G_2$.*
(3) *P is a parallelogram.*

For a quadrangle $ABCD$, we put as follows:

$$AB = l_1,\ BC = l_2,\ CD = l_3,\ DA = l_4. \tag{1.1}$$

In order to study the relationships between the centroid G_1 and the centroid G_2 of a convex quadrangle, for the intersection point M of the two diagonals AC and BD we define as follows:

$$\triangle ABM = m_1, \quad \triangle BCM = m_2, \quad \triangle CDM = m_3, \quad \triangle DAM = m_4. \tag{1.2}$$

The perimeter l and the area m of the convex quadrangle $ABCD$ are respectively given by

$$l = l_1 + l_2 + l_3 + l_4 \tag{1.3}$$

and

$$m = m_1 + m_2 + m_3 + m_4. \tag{1.4}$$

Next, using the above notations, in Section 4 we establish a characterization theorem for convex quadrangles satisfying $G_1 = G_2$ as follows.

Theorem B. *Let P denote a convex quadrangle $ABCD$. Then the following are equivalent.*

(1) *P satisfies $G_1 = G_2$.*
(2) *P satisfies both*

$$l(m_3 + m_4) = m\left(3(l_3 + l_4) - l\right) \tag{1.5}$$

and

$$l(m_1 + m_4) = m\left(3(l_1 + l_4) - l\right). \tag{1.6}$$

Finally, in Section 5 we give some examples of quadrangles which are not parallelograms but satisfy $G_1 = G_2$ as follows.

Example C. There exist quadrangles (convex or concave) which are not parallelograms but satisfy $G_1 = G_2$.

For further study, we raise a question as follows:

Question D. 1) Which quadrangles satisfy $G_1 = G_2$?

2) Which pentagons (or generally n-gons) satisfy $G_0 = G_1 = G_2$?

For finding the centroid G_2 of all types of convex and concave polygons, we refer [3]. In [10], mathematical definitions of centroid G_2 of planar bounded domains were given. It was shown that the centroid G_0 of the vertices of a simplex in an n-dimensional space always coincides with the centroid G_n of the simplex ([11]).

Archimedes discovered and proved the area properties of parabolic sections and then formulated the centroid of parabolic sections ([12]). Some characterizations of parabolas using these properties were given in [5, 8, 9]. Furthermore, Archimedes also proved the volume properties of the region surrounded by a paraboloid of rotation and a plane ([12]). For characterizations of elliptic paraboloid or ellipsoids with respect to these volume properties, we refer [4, 6, 7].

2. Preliminaries

Let us consider four distinct points A, B, C and D in the plane $\mathbb{R}^2$. We say they determine the quadrangle $ABCD$ if they satisfy the following conditions.

(C1) The union of four successive segments $\{AB, BC, CD, DA\}$ bounds a simply connected domain in the plane $\mathbb{R}^2$.

(C2) Three points of them are not collinear.

If P denotes the quadrangle $ABCD$, then the four points A, B, C and D are called the vertices of P, the four successive segments the edges of P and the segments AC and BD the diagonals of P, respectively.

For a quadrangle $ABCD$, we have the following, where we use the notations given in (1.1), (1.3) and (1.4).

Proposition 2.1. *Let P denote the quadrangle $ABCD$. Then we have the following.*

(1) *The centroid G_0 of P is given by*

$$G_0 = \frac{A + B + C + D}{4}. \tag{2.1}$$

(2) *The centroid G_1 of P is given by*

$$G_1 = \frac{(l_4 + l_1)A + (l_1 + l_2)B + (l_2 + l_3)C + (l_3 + l_4)D}{2l}. \tag{2.2}$$

(3) *If $m = \delta \pm \beta$, where $\delta = \triangle ABC$ and $\beta = \triangle ACD$, then the centroid G_2 of P is given by*

$$G_2 = \frac{mA + \delta B + mC \pm \beta D}{3m}, \tag{2.3}$$

Proof. It is trivial to prove (1). It is straightforward to prove (2), or see [3].

For (3), we prove only the case $m = \delta - \beta$. In this case, the vertex D of P lies in the interior of the triangle ABC. Note that the disjoint union of the interior of ACD and the interior of quadrangle P becomes the interior of ABC

excepts a measure zero set. Thus we get the following:

$$\frac{mG_2 + \delta\left(\frac{A+C+D}{3}\right)}{m+\delta} = \frac{A+B+C}{3}, \tag{2.4}$$

which shows that (2.3) holds.

The remaining cases can be treated similarly. This completes the proof of Proposition 2.1. □

It is trivial to show the following.

Proposition 2.2. *The centroids G_0, G_1 and G_2 of a parallelogram P coincide with the intersection point G of two diagonals of the parallelogram P.*

In the proof of Theorem A, we need the following proposition which can be proved easily.

Proposition 2.3. *Let P denote the quadrangle $ABCD$. Then the diagonals AC and BD of P are not parallel to each other.*

Finally, we give an example which shows the necessity of condition (C1).

Example 2.4. We consider four points $A(1,0), B(1,1), C(2,1)$ and $D(0,0)$. Then the centroid G_0 of the four points and the centroid G_1 of four successive segments coincide with $G_0 = G_1 = (1, 1/2)$. But they does not satisfy the condition (C1).

3. Characterizations of parallelograms

In this section, we prove Theorem A stated in Section 1.

Suppose that a quadrangle $ABCD$ denoted by P satisfies $G_0 = G_1$. Then Proposition 2.1 shows that

$$l(A+B+C+D)=2(l_4+l_1)A+2(l_1+l_2)B+2(l_2+l_3)C+2(l_3+l_4)D, \tag{3.1}$$

where we use (1.1) and (1.3). By a translation, we may assume that the point D is the origin. Then (3.1) becomes

$$yB = x(C - A), \tag{3.2}$$

where we put

$$x = l_2 + l_3 - l_1 - l_4, \quad y = l_3 + l_4 - l_1 - l_2. \tag{3.3}$$

If $y \neq 0$, then (3.2) implies that the two diagonals DB and AC of quadrangle P are parallel to each other. This contradiction shows that $y = 0$, and hence from (3.2) we also have $x = 0$. It follows from (3.3) that $l_1 = l_3$ and $l_2 = l_4$. This completes the proof of (1) ⇒ (3) in Theorem A.

Now, suppose that a quadrangle $ABCD$ denoted by P satisfies $G_0 = G_2$.

We, first of all, claim that the quadrangle P is convex. Otherwise, a vertex (say, A) lies in the interior of the triangle BCD. If we put $\delta = \triangle ABC$ and $\beta = \triangle ACD$, then we have $m = \delta + \beta$. Hence Proposition 2.1 yields that

$$3(\beta+\delta)(A+B+C+D) = 4(\beta+\delta)A + 4\delta B + 4(\beta+\delta)C + 4\beta D. \tag{3.4}$$

By a translation, we may assume that the point D is the origin. Then (3.4) becomes

$$(3\beta - \delta)B = (\beta + \delta)(A + C). \tag{3.5}$$

If $3\beta - \delta = 0$, then (3.5) shows that $A + C = 0$ because $\beta + \delta > 0$. This shows that A, C and $D(= 0)$ are collinear. This contradiction yields $3\beta - \delta \neq 0$

Suppose that $3\beta - \delta > 0$. Then (3.5) shows that B is a positive multiple of $A + C$, and hence the quadrangle P is a convex quadrangle, which is a contradiction.

Suppose that $3\beta - \delta < 0$. Then it follows from (3.5) that B is a negative multiple of $A+C$, and hence the point $D(= 0)$ lies in the interior of the triangle ABC, which is a contradiction.

The above contradictions all together imply that the quadrangle P is a convex quadrangle with $3\beta - \delta > 0$. Obviously, P satisfies (3.5). We put $E = A + C$. Then, it follows from (3.5) that the vertex B lies on the diagonal DE of the parallelogram $AECD$. Hence we get for the intersection point M of diagonals DE and AC of the parallelogram $AECD$

$$\triangle ABM = \triangle BCM(= \frac{\delta}{2}), \quad \triangle CDM = \triangle DAM(= \frac{\beta}{2}), \tag{3.6}$$

where we use $AM = CM$. This shows that

$$\triangle ABD = \triangle BCD(= \frac{m}{2}). \tag{3.7}$$

Now, we repeat the similar argument as in the above. Then, by letting $\gamma = \triangle ABD$ and $\alpha = \triangle BCD$ with $m = \gamma + \alpha$, we may prove that

$$\triangle ABC = \triangle ACD(= \frac{m}{2}), \tag{3.8}$$

which shows that $\beta = \delta$. Hence it follows from (3.5) that $B = A + C$. This completes the proof of (2) $\Rightarrow$ (3) in Theorem A.

Conversely, both (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) follow from Proposition 2.2. This completes the proof of Theorem A.

4. Quadrangles satisfying $G_1 = G_2$

In this section, using the notations in (1.1)-(1.4), we prove Theorem B stated in Section 1.

We consider a convex quadrangle $ABCD$ denoted by P. Then Proposition 2.1 shows that the centroid G_1 of P is given by

$$G_1 = \frac{(l_4 + l_1)A + (l_1 + l_2)B + (l_2 + l_3)C + (l_3 + l_4)D}{2l}. \tag{4.1}$$

If we let $m = \delta + \beta$, where $\delta = \triangle ABC = m_1 + m_2$ and $\beta = \triangle ACD = m_3 + m_4$, then by Proposition 2.1 the centroid G_2 of P is given by

$$G_2 = \frac{mA + (m_1 + m_2)B + mC + (m_3 + m_4)D}{3m}. \tag{4.2}$$

By letting $\gamma = \triangle ABD = m_1+m_4$ and $\alpha = \triangle BCD = m_2+m_3$ with $m = \gamma+\alpha$, we also get from Proposition 2.1

$$G_2 = \frac{(m_1+m_4)A + mB + (m_2+m_3)C + mD}{3m}. \tag{4.3}$$

It follows from (4.2) and (4.3) that we always have

$$(m_2+m_3)A + (m_1+m_4)C = (m_3+m_4)B + (m_1+m_4)D. \tag{4.4}$$

Now, suppose that the quadrangle P satisfies $G_1 = G_2$. By a translation of P, we may assume that the vertex D is the origin. Then from (4.4) we obtain

$$B = \frac{m_2+m_3}{m_3+m_4}A + \frac{m_1+m_4}{m_3+m_4}C. \tag{4.5}$$

Together with the assumption $G_1 = G_2$, (4.1) and (4.3) show that

$$B = \frac{2l(m_1+m_4) - 3m(l_1+l_4)}{m\,(3(l_1+l_2)-2l)}A + \frac{2l(m_2+m_3) - 3m(l_2+l_3)}{m\,(3(l_1+l_2)-2l)}C. \tag{4.6}$$

Since the vectors $A(= A-D)$ and $C(= C-D)$ are linearly independent, the coefficients of A (resp., C) in (4.5) and (4.6) are equal to each other. Hence, by adding the coefficients in (4.5) and (4.6), respectively, we obtain

$$\frac{m}{m_3+m_4} = \frac{l}{3(l_3+l_4)-l}. \tag{4.7}$$

This shows that (1.5) holds.

In order to prove (1.6), we translate the quadrangle P so that the vertex A is the origin. Then from (4.4) we obtain

$$C = \frac{m_3+m_4}{m_1+m_4}B + \frac{m_1+m_2}{m_1+m_4}D. \tag{4.8}$$

Together with the assumption $G_1 = G_2$, (4.1) and (4.2) show that

$$C = \frac{2l(m_1+m_2) - 3m(l_1+l_2)}{m\,(3(l_2+l_3)-2l)}B + \frac{2l(m_3+m_4) - 3m(l_3+l_4)}{m\,(3(l_2+l_3)-2l)}D. \tag{4.9}$$

The same argument as in the above shows that

$$\frac{m}{m_1+m_4} = \frac{l}{3(l_1+l_4)-l}, \tag{4.10}$$

which implies that (1.6) holds.

This completes the proof of (1) $\Rightarrow$ (2) in Theorem B.

Conversely, suppose that the quadrangle P satisfies both (1.5) and (1.6). Then, first note that together with (1.3) and (1.4), (1.5) and (1.6) respectively imply

$$l(m_1+m_2) = m\,(3(l_1+l_2)-l) \tag{4.11}$$

and

$$l(m_2+m_3) = m\,(3(l_2+l_3)-l)\,. \tag{4.12}$$

Now, we translate the quadrangle P so that the vertex D is the origin. Then from (4.4) we see that (4.5) always holds. Using (1.6), (4.11) and (4.12), it follows from (4.1) that

(4.13)
$$G_1 = \frac{1}{6m}\left((m+m_1+m_4)A + (m+m_1+m_2)B + (m+m_2+m_3)C\right).$$

Replacing B in (4.13) with that in (4.5), we get

$$G_1 = \frac{1}{6m(m_3+m_4)}(xA+yC), \tag{4.14}$$

where we put

$$x = (m_3+m_4)(m+m_1+m_4) + (m_2+m_3)(m+m_1+m_2) \tag{4.15}$$

and

$$y = (m_1+m_4)(m+m_1+m_2) + (m_3+m_4)(m+m_2+m_3). \tag{4.16}$$

On the other hands, it follows from (4.3) and (4.5) that

$$G_2 = \frac{1}{3m(m_3+m_4)}(zA+wC), \tag{4.17}$$

where we use

$$z = (m_1+m_4)(m_3+m_4) + m(m_2+m_3) \tag{4.18}$$

and

$$w = m(m_1+m_4) + (m_2+m_3)(m_3+m_4). \tag{4.19}$$

Finally, it is easy to show that

$$x - 2z = 0, \quad y - 2w = 0, \tag{4.20}$$

which, together with (4.14) and (4.17), implies that P satisfies $G_1 = G_2$. This yields that (2) $\Rightarrow$ (1) holds. Therefore the proof of Theorem B is completed.

5. Examples

In this section, we prove Example C stated in Section 1.

We consider the four points in the plane $\mathbb{R}^2$ defined by

$$A(x,0), B(0,1), C(-1,0), D(0,-1). \tag{5.1}$$

If $x > 0$, then the quadrangle $ABCD$ is convex. In case $x < 0$ with $x \neq -1$, it is concave. We denote by $P(x)$ the quadrangle $ABCD$. Then for $x > 0$ the centroids G_1 and G_2 of $P(x)$ are respectively given by

$$G_1 = \left(\frac{x\sqrt{x^2+1}-\sqrt{2}}{2\left(\sqrt{x^2+1}+\sqrt{2}\right)}, 0\right) \tag{5.2}$$

and

$$G_2 = \left(\frac{x-1}{3}, 0\right). \tag{5.3}$$

Note that even if $x < 0$ with $x \neq -1$, the centroids G_1 and G_2 of $P(x)$ are also given by (5.2) and (5.3), respectively.

It follows from (5.2) and (5.3) that $P(x)$ satisfies $G_1 = G_2$ if and only if

$$f(x) = g(x), \tag{5.4}$$

where we put

$$f(x) = (x+2)\sqrt{x^2+1}, \quad g(x) = \sqrt{2}(2x+1). \tag{5.5}$$

When $x \geq 0$, note that

$$f(0) = 2 > \sqrt{2} = g(0), \quad f(1) = 3\sqrt{2} = g(1) \tag{5.6}$$

and

$$f'(1) = \frac{5\sqrt{2}}{2} > 2\sqrt{2} = g'(1). \tag{5.7}$$

Hence, there exists a number $a_1 \in (0,1)$ which satisfies $f(a_1) = g(a_1)$. Thus, the convex quadrangle $P(a_1)$ satisfies $G_1 = G_2$ but it is not a parallelogram.

When $x < 0$ with $x \neq -1$, note that

$$f(-2) = 0 > -3\sqrt{2} = g(-2). \tag{5.8}$$

If $x < -2$, we have $f(x) < 0$ and $g(x) < 0$. Furthermore, the functions satisfy

$$\lim_{x \to -\infty} \frac{f(x)}{g(x)} = \infty. \tag{5.9}$$

Hence, there exists a number $b(< -2)$ such that $f(b) < g(b)$. Thus, there exists also a number $a_2 \in (b, -2)$ satisfying $f(a_2) = g(a_2)$. Therefore the concave quadrangle $P(a_2)$ satisfies $G_1 = G_2$ but it is not a parallelogram.

This completes the proof of Example C.

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