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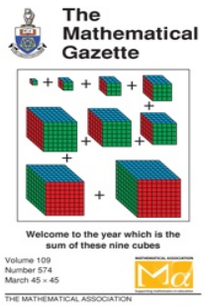

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Constructions of equi-inclined lines in cyclic and tangential quadrilaterals

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ABSTRACT

In a cyclic quadrilateral, the perpendicular bisectors of the sides are concurrent at the circumcentre. Similarly, in a tangential quadrilateral, the bisectors of the interior angles are concurrent at the incentre. Thus one way of thinking of possible generalisations is to take equi-inclined lines (for a more precise explanation see below) instead of perpendicular bisectors (cyclic quadrilaterals) and angle bisectors (tangential quadrilaterals). To ask and answer such 'what if?' questions (What if the bisectors are not perpendicular but equi-inclined? What if we take equi-inclined lines instead of angle bisectors?) is very common in the process of doing mathematics. Asking such questions is a crucial part of posing problems (see [1, 2]), and an important resource, one could even say a 'motor', for providing new results and insights in mathematics. In this paper we will explore the above-mentioned 'what if?' questions.

KEYWORDS

Euclidean geometry; cyclic quadrilaterals; tangential quadrilaterals; equi-inclined lines

1. Equi-inclined bisectors of a cyclic quadrilateral

Cyclic quadrilaterals are ones having a circumcircle, such as *ABCD* in Figure 1. In most high school geometry courses around the

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world, one of the first theorems of circle geometry that students will encounter is the following: the perpendicular bisector of a circle's chord passes through the centre of the circle. For the cyclic quadrilateral $ABCD$ shown in Figure 1, this implies that the perpendicular bisectors of its four sides pass through the circumcentre, i.e. are concurrent (at O).

In this section we will consider a special generalisation of perpendicular bisectors of cyclic quadrilaterals, namely, to stay with cyclic quadrilaterals but to drop the property 'perpendicular' from the bisectors. In other words, we will investigate what happens if we replace the 90°angle at the midpoints of the sides of a cyclic quadrilateral by another constant angle φ? Such 'what if?' questions can readily be explored by using dynamic geometry to see experimentally what happens. And, indeed, our experiment produces an interesting result (see Figure 1).

Let us now explain more precisely what we mean by the term *equi-inclined lines* (see Figure 1). Let E, F, G, H be the respective midpoints of the sides of the cyclic quadrilateral $ABCD$; then construct *equi-inclined* lines through these points yielding quadrilateral $A'B'C'D'$ in the following way: the oriented angles between

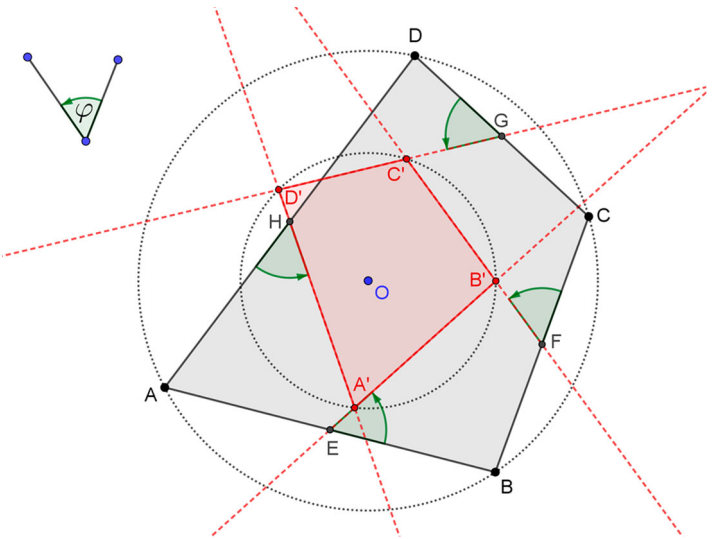

FIGURE 1. Equi-inclined lines through the midpoints of the sides of a cyclic quadrilateral.

the constructed lines and the respective sides of the quadrilateral are all equal, say $\varphi \in [0^\circ, 180^\circ)$; then the equi-inclined lines of the edges DA and AB meet at A', the equi-inclined lines of AB and BC meet at B', the equi-inclined lines of BC and CD meet at C', and the equi-inclined lines of CD and DA meet at D' (the case $\varphi = 90^\circ$ would correspond to perpendicular bisectors).

From our dynamic construction as shown in Figure 1, the following conjectures immediately arose and were quickly confirmed by dragging to check a number of cases: $A'B'C'D'$ seems to be cyclic as well, $A'B'C'D'$ seems to be similar to $ABCD$, and $A'B'C'D'$ and $ABCD$ seem to be concentric (meaning they share a common circumcentre O).

These observations we now formulate as a theorem.

Theorem 1: *Let ABCD be an arbitrary cyclic quadrilateral (also crossed is allowed) with midpoints of the sides E, F, G, H. Construct through these midpoints equi-inclined lines to the respective sides, say with angle* $\varphi \neq 90^\circ$*, yielding* $A'B'C'D'$ *(for* $\varphi = 90^\circ$ *the vertices* A'*,* B'*,* C'*,* D' *would all coincide at the circumcentre O of ABCD). Then* $A'B'C'D'$ *is also cyclic, similar to ABCD, and both quadrilaterals share the common circumcentre O, and the similarity factor is* $|\cos\varphi|$.

Proof*[1]*: In Figure 2, one can immediately see that quadrilateral $AEA'H$ is cyclic since the interior angles at E and H are supplementary. Thus also the angles $\angle EAH = \angle BAD$ and $\angle HA'E$ are supplementary, yielding $\angle A = \angle A'$ for the corresponding interior angles in the quadrilaterals $ABCD$ and $A'B'C'D'$. Analogously we can establish $\angle B = \angle B'$, $\angle C = \angle C'$ and $\angle D = \angle D'$, telling us that corresponding interior angles in $ABCD$ and $A'B'C'D'$ are equal (note that this equality of angles $\angle A = \angle A'$, ..., $\angle D = \angle D'$ is always the case, as it is for general, not cyclic, quadrilaterals $ABCD$). But this equality of angles does not suffice to conclude that $ABCD$ and $A'B'C'D'$ are similar.

We will show that there is a spiral similarity,[2] centred at O (circumcentre of $ABCD$) with angle φ and factor $|\cos\varphi|$ which takes $ABCD$ to $A'B'C'D'$ (see Figure 2).

In Figure 2, quadrilateral $AEOH$ is cyclic also, due to the homothety centred at A, factor ½, applied to the circumcircle of $ABCD$,

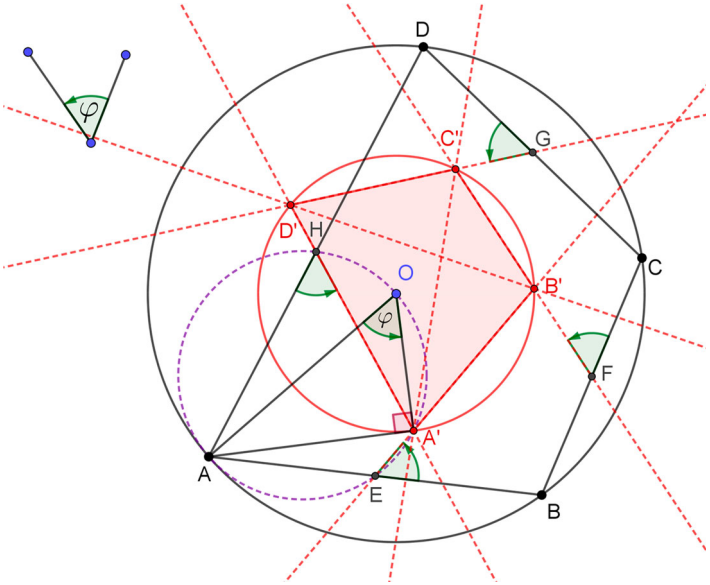

FIGURE 2. To the proof of Theorem 1a.

and AO is a diameter of the circumcircle of $AEOH$. Thus A, E, A', O, H are five concyclic points. Moreover, $\angle AHA' = \varphi = \angle AOA'$ and $\angle OA'A = 90^\circ$. Analogous working, with the vertices B, C, D instead of A, yields four congruent right triangles $\Delta AA'O \cong \Delta BB'O \cong \Delta CC'O \cong \Delta DD'O$ (in order not to overload Figure 2 we show $\Delta AA'O$), all of them having interior angle φ at vertex O, which is the constant rotation angle of the spiral similarity we have in mind. Thus $\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \frac{OD'}{OD} = |\cos\varphi|$, which is the constant lengths ratio of the spiral similarity and completes the proof.

Additional remarks:

- From the first part of the above proof it is clear: if $ABCD$ is not cyclic, the equality of corresponding angles $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, and $\angle D = \angle D'$ still holds, but the similarity property no longer holds.
- The quadrilateral $A'B'C'D'$ is biggest for $\varphi = 0^\circ$ (in this case $A'B'C'D'$ coincides with $ABCD$ itself); it is smallest for $\varphi = 90^\circ$ (in this case degenerating to a single point, namely O).

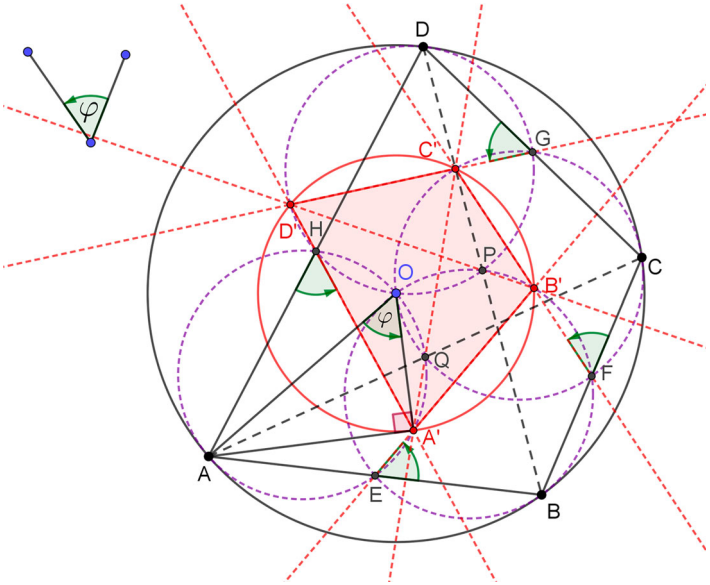

FIGURE 3. P and Q must lie on special circles.

- Let Q be the intersection point of the two diagonals AC and $A'C'$. Then due to the above-mentioned spiral similarity, we have $\angle AQA' = \varphi = \angle CQC'$ (see Figure 3). Thus Q must also lie on circles $AEA'OH$ and $CGC'OF$ (note that $\angle AOA' = \varphi = \angle COC'$; analogously with P, diagonals BD and $B'D'$, and the two other circles).
- At least, a sort of 'weak converse' holds: $A'B'C'D'$ is cyclic *only* if $ABCD$ is cyclic. This is clear, since the angles at the corresponding vertices must be equal: $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$ and $\angle D = \angle D'$.

With this knowledge it is easy to investigate a closely related sort of equi-inclined lines, namely lines equi-inclined to so-called *maltitudes*, see below. Above we used equi-inclined lines to the *sides* of the cyclic quadrilateral through the respective midpoints, which is equivalent to say that they are equi-inclined to the perpendicular bisectors of the sides (this changes the inclination angle just by 90°). *Maltitudes* of a quadrilateral are lines through the midpoints of the sides perpendicular to the respective opposite sides, and in case of

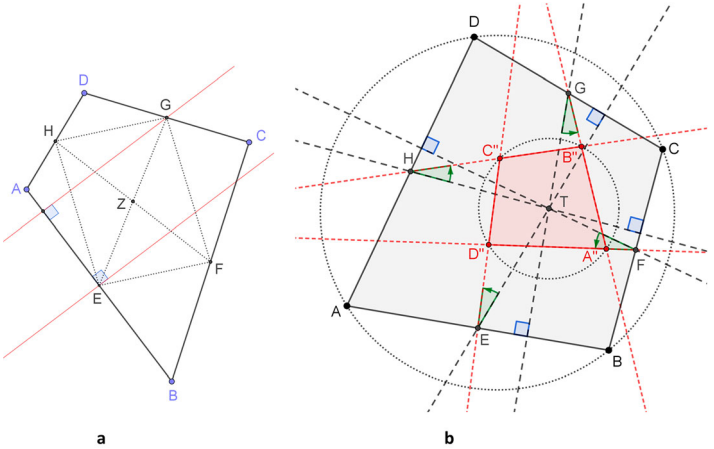

FIGURE 4. (a) Maltitudes and perpendicular bisectors are half turns of each other about Z. (b) To Theorem 1b.

a cyclic quadrilateral these maltitudes are concurrent (at the so-called *anticentre*, which is the circumcentre reflected in the vertex centroid of the cyclic quadrilateral, see [4, 5]). This is clear because maltitudes can be seen as the image of the perpendicular bisectors of the sides under a half turn centred at the vertex centroid Z of the quadrilateral (see Figure 4a).

Thus, by the half turn symmetry centred at the vertex centroid Z, properties of the perpendicular bisectors are transferred to properties of the maltitudes, e.g. their concurrency at a special point (circumcentre, anticentre; these two lie symmetric to Z). Also, properties concerning equi-inclination to perpendicular bisectors transfer naturally to properties of equi-inclination to maltitudes. In our context this means the following theorem.

Theorem 2: *Let ABCD be an arbitrary cyclic quadrilateral (also crossed is allowed) with midpoints of the sides E, F, G, H. Construct through these midpoints equi-inclined lines to the maltitudes, say with angle* $\varphi \neq 0°$, *yielding* $A''B''C''D''$ *(for* $\varphi = 0°$ *the vertices* A'', B'', C'', D'' *would all coincide at the anticentre T of ABCD). Then* $A''B''C''D''$ *is also cyclic, similar to ABCD, has its circumcentre at the anticentre T of ABCD, and the similarity factor is* $|\sin\varphi|$ *(see Figure 4b).*

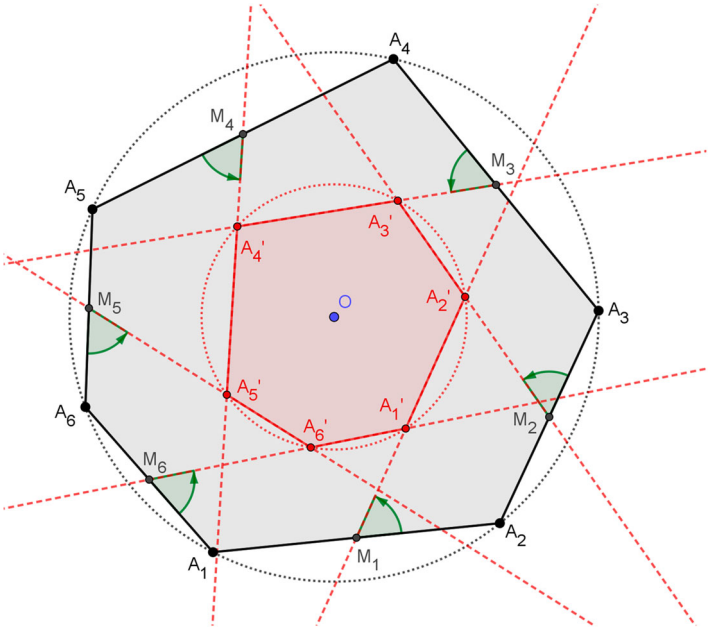

FIGURE 5. To Theorem 2.

Now, we ask another 'what if?' question, namely, what if the cyclic polygon has more than four vertices? It is easy to see from the above similarity proof that Theorem 2 immediately generalises to any cyclic polygon. We formulate this as Theorem 3.

Theorem 2: *Let $A_1 \ldots A_n$ be an arbitrary cyclic polygon (including crossed ones) with respective midpoints of the sides $M_1, \ldots, M_n$. Construct at these midpoints equi-inclined lines to the respective sides, say with angle $\varphi \neq 90^\circ$, yielding the polygon $A_1' \ldots A_n'$. Then $A_1' \ldots A_n'$ is also cyclic, similar to $A_1 \ldots A_n$, both polygons share the common circumcentre O, and the similarity factor is $|\cos \varphi|$ (Figure 5 shows an example with $n = 6$, a hexagon, and $\varphi = 60^\circ$, yielding similarity factor ½).*

A dynamic online version for exploring Theorems 1 and 3 (pentagon) is available at http://dynamicmathematicslearning.com/equi-inclined-bisectors-of-cyclic-quad.html.

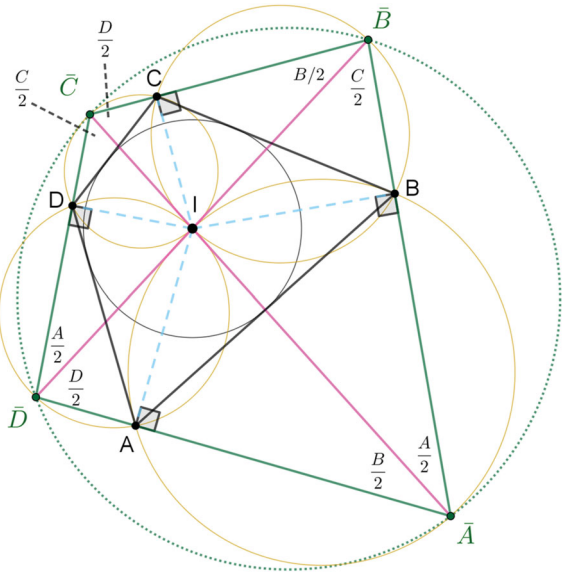

FIGURE 6. To Lemma 1.

2. Equi-inclined lines in a tangential quadrilateral

Tangential quadrilaterals are ones where the four sides of the quadrilateral (or their extensions) touch a common circle. The most common case is where the quadrilateral $ABCD$ is convex, then the circle is commonly called its *incircle* (Figure 6). Since the incircle touches the four sides of $ABCD$, the distances from the incentre to the four sides (the radii) are equal. And since an angle bisector is the locus of all points equi-distant from the two rays forming the angle, it follows that all four angle bisectors must pass through the incentre, i.e. are concurrent (at I).

In this section, we will consider the analogous question to Section 1 when taking tangential quadrilaterals and another sort of equi-inclination, namely lines through the vertices, equi-inclined regarding the corresponding angle bisectors. We will see that many issues will work analogously, proving once more that cyclic quadrilaterals and tangential quadrilaterals are — *mutatis mutandis* — very closely related figures. Like in Section 1, a spiral similarity will play a crucial role in establishing a proof.

In problem solving it is often useful, as pointed out by Polya [6], to try and relate a new or unknown problem to a known one. In the following proofs we are relating the equi-inclined result for a tangential quadrilateral to our first result for a cyclic quadrilateral by means of a construction and the following lemma.

Lemma: *Given an arbitrary tangential quadrilateral ABCD with incentre I. Construct perpendiculars*[3] *to IA, IB, IC, ID at the vertices A, B, C, D yielding the intersection points* $\bar{A}, \bar{B}, \bar{C}, \bar{D}$. *Then the following two facts hold:*

1. $\bar{A}\bar{B}\bar{C}\bar{D}$ *is cyclic.*
2. *The quadrilaterals* $A\bar{A}BI$, $B\bar{B}CI$, $C\bar{C}DI$, $D\bar{D}AI$ *are cyclic, having* $I\bar{A}, I\bar{B}, I\bar{C}, I\bar{D}$ *as diameters of the respective circumcircles (see Figure 6; note that this fact can easily be generalised to any tangential n-sided polygon* $A_1 \ldots A_n$*).*

Proof 4: By the converse of Thales' theorem,[5] from the right angles at A, B, C, D, we can conclude that the quadrilaterals $A\bar{A}BI$, $B\bar{B}CI$, $C\bar{C}DI$, $D\bar{D}AI$ are cyclic and $I\bar{A}, I\bar{B}, I\bar{C}, I\bar{D}$ are diameters of the respective circumcircles. By the inscribed angle theorem,[6] the angles $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}, \frac{D}{2}$ are parts of the angles $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, as shown in Figure 6. This yields immediately $\bar{A} + \bar{C} = \frac{A+B+C+D}{2} = 180^\circ$ (note that $A + B + C + D = 360^\circ$) which makes $\bar{A}\bar{B}\bar{C}\bar{D}$ cyclic.

Additional remarks:

- $\angle\bar{A}I\bar{B} + \angle\bar{B}I\bar{C} = \left(180^\circ - \frac{A}{2} - \frac{C}{2}\right) + \left(180^\circ - \frac{B}{2} - \frac{D}{2}\right) = 180^\circ$, telling us that I lies on $\bar{A}\bar{C}$. Analogously with the diagonal $\bar{B}\bar{D}$; for another proof see e.g. [3, p. 5, Theorem 3].
- 'Opposite' circles $AID\bar{D}$ and $CIB\bar{B}$ touch each other at I, analogously with the 'opposite' circles $BIA\bar{A}$ and $DIC\bar{C}$ (note that $I\bar{A}, I\bar{B}, I\bar{C}, I\bar{D}$ are diameters of the respective circles).

Now we come to another but equivalent interpretation of (equi-inclination) of lines. Let $ABCD$ be a tangential quad with incentre I. We construct equi-inclined lines through the vertices A, B, C,

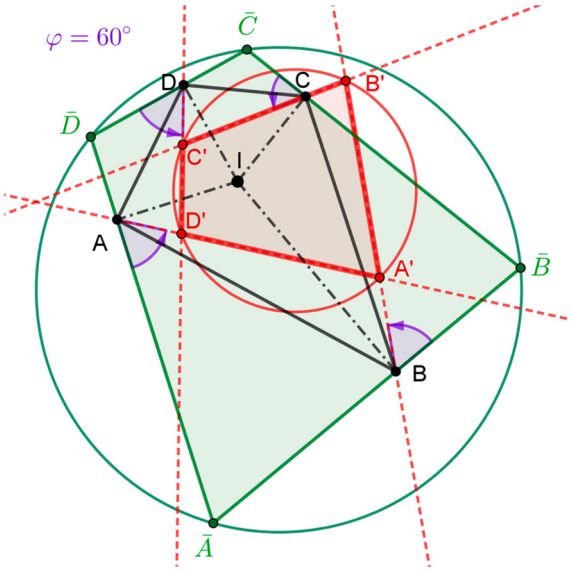

FIGURE 7. Equi-inclined lines through the vertices of a tangential quadrilateral.

D as follows: for convenience, the equi-inclination we use here is not defined directly by the oriented angle of the lines to the bisectors of the interior angles themselves but to their perpendiculars, and denote this angle by $\varphi \in [0°, 180°]$, i.e. the angle to the sides of the quadrilateral $\bar{A}\bar{B}\bar{C}\bar{D}$ mentioned in Lemma. These equi-inclined lines establish, as above, the quadrilateral $A'B'C'D'$, see Figure 7 where, as an example, $\varphi = 60°$.

Looking at Figure 7, the following conjectures were found by using dynamic geometry: quadrilateral $A'B'C'D'$ seems to be cyclic and similar to $\bar{A}\bar{B}\bar{C}\bar{D}$, and the similarity factor seems to be ½ (which is, actually, $|\cos\varphi|$ for $\varphi = 60°$). This is quite analogous to Section 1, but here the circumcircles are not concentric, as they were there.

This we formulate as a theorem.

Theorem 4: *Let ABCD be an arbitrary tangential quadrilateral. Construct the corresponding quadrilateral* $\bar{A}\bar{B}\bar{C}\bar{D}$ *as in Lemma 1 (Figures 6 and 7) and through A, B, C, D equi-inclined lines to the sides of* $\bar{A}\bar{B}\bar{C}\bar{D}$*, say with an inclination angle* $\varphi \neq 90°$*, yielding* $A'B'C'D'$ *(for* $\varphi = 90°$ *the vertices* A'*,* B'*,* C'*,* D' *would all coincide at*

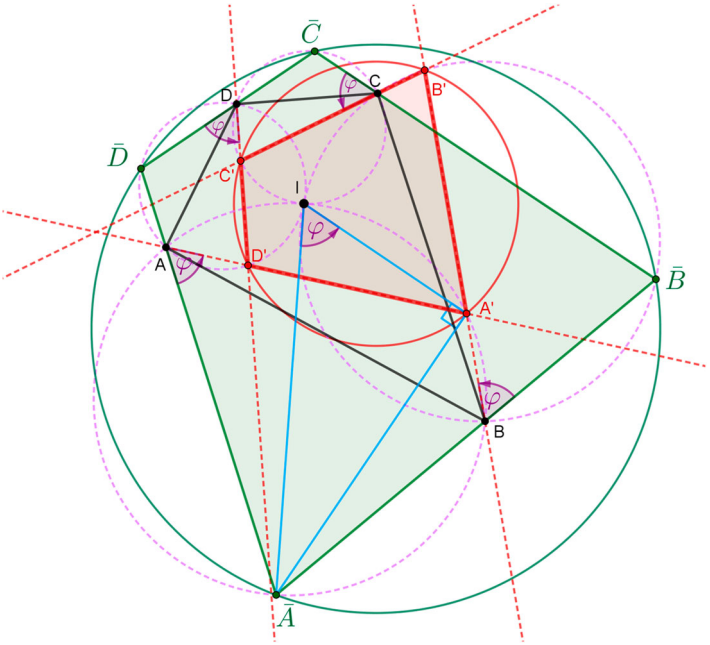

FIGURE 8. Top the proof of Theorem 3.

the incentre I of ABCD). Then $A'B'C'D'$ is cyclic, similar to $\bar{A}\bar{B}\bar{C}\bar{D}$, and the similarity factor is $|\cos\varphi|$.

Proof 7: We will show that there is a spiral similarity, centred at I (incentre of $ABCD$) with angle φ and factor $|\cos\varphi|$ which takes $\bar{A}\bar{B}\bar{C}\bar{D}$ to $A'B'C'D'$ (Figure 8).

From the Lemma, we know that the quadrilaterals $A\bar{A}BI$, $B\bar{B}CI$, $C\bar{C}DI$, $D\bar{D}AI$ are cyclic.

Furthermore, $\bar{A}BA'A$ is cyclic because the interior angles at B and A are supplementary, yielding the five concyclic points $\bar{A}, B, A', I, A$. This enables us to conclude that $\angle\bar{A}IA' = \varphi$, and since $\bar{A}I$ is a diameter, we know $\angle IA'\bar{A} = 90°$, in other words A' is the image of $\bar{A}$ under the spiral similarity centred at I with angle φ and factor $|\cos\varphi|$. In an analogous fashion, we can show $\bar{B} \mapsto B'$, $\bar{C} \mapsto C'$, $\bar{D} \mapsto D'$ under the mentioned spiral similarity, which completes the proof (this time the triangles

$\Delta I\bar{A}A'$, $\Delta I\bar{B}B'$, $\Delta I\bar{C}C'$, $\Delta I\bar{D}D'$ are not congruent, they are only similar; in Section 1 the corresponding triangles were congruent).

A dynamic online version for exploring Theorem 3 is available at http://dynamicmathematicslearning.com/equi-inclined-angle-partition-tangential-quad.html.

Additional remarks:

- The following is easy to see (proof left to the reader): If $ABCD$ is not tangential, $A'B'C'D'$ is still cyclic, but the similarity property no longer holds.
- The quadrilateral $A'B'C'D'$ is biggest for $\varphi = 0^\circ$ (in this case A', B', C', D' coincide with the corresponding points $\bar{A}, \bar{B}, \bar{C}, \bar{D}$); it is smallest for $\varphi = 90^\circ$ (in this case degenerating to a single point, namely I, the incentre of $ABCD$).
- Let O' be the circumcentre of $A'B'C'D'$ and $\bar{O}$ the circumcentre of $\bar{A}\bar{B}\bar{C}\bar{D}$. If one lets $\varphi \in [0^\circ, 180^\circ)$ go through all possible values and observes the corresponding trace of O', one can see that the locus of O' is a circle with diameter $I\bar{O}$ (see Figure 9). Using the converse of Thales' theorem and the above-mentioned spiral similarity, this can readily be seen from the right triangle $I\bar{O}O'$ with $\angle\bar{O}IO' = \varphi$ and $IO' = I\bar{O} \cdot |\cos\varphi|$.

Now we come to a final 'what if?' question, namely, what if the tangential polygon has more than four vertices? It is easy to see that Theorem 4, in a way, generalises to any n-sided tangential polygon. The only difference is that the respective polygons, formed on the one hand by the perpendiculars at the vertices to the angle bisectors, and on the other hand by the equi-inclined lines, are no longer cyclic for $n > 4$. We formulate this as Theorem 5.

Theorem 5: *Let $A_1 \ldots A_n$ be an arbitrary tangential n-sided polygon with $n \geq 3$. Construct equi-inclined lines to the perpendiculars of the angle bisectors at respective vertices, say with an inclination angle $\varphi \neq 90^\circ$, yielding $A_1' \ldots A_n'$ (for $\varphi = 90^\circ$ the vertices $A_1', \ldots, A_n'$ would all coincide at the incentre I of $A_1 \ldots A_n$). Then $A_1' \ldots A_n'$ is similar to $\bar{A}_1 \ldots \bar{A}_n$ (in Figure 10 we have $n = 6$; defined like $\bar{A}\bar{B}\bar{C}\bar{D}$ in the Lemma and Figure 6 for the case of $n = 4$), and the similarity factor is $|\cos\varphi|$. More precisely, there is a spiral similarity centred at the incentre I of $A_1 \ldots A_n$ with similarity factor $|\cos\varphi|$ which maps*

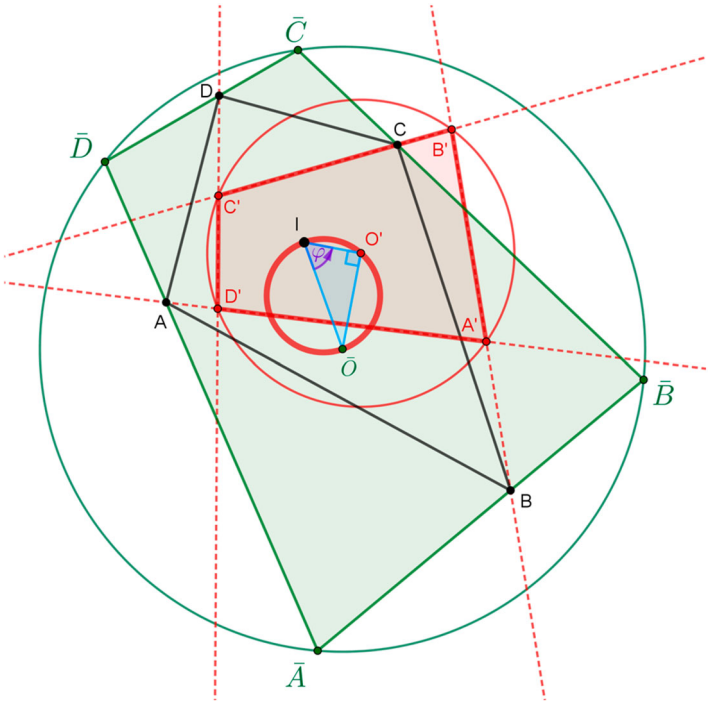

FIGURE 9. The locus of O′.

$\bar{A}_1 \ldots \bar{A}_n \mapsto A_1' \ldots A_n'$ *(see Figure 10 an example with* $\varphi = 60°$ *and thus similarity factor ½).*

Proof: Since the proof of Theorem 4 concerning the spiral similarity does not depend on the number of vertices, Theorem 5 is immediate from the proof of Theorem 4.

The only difference to Theorem 4 is that the polygons $\bar{A}_1 \ldots \bar{A}_n$ and $A_1' \ldots A_n'$ are, in general, not cyclic any more for $n > 4$. We did not find any reference to the theorems presented in this paper, so maybe they are new (or well-hidden in the literature). If a reader happens to know corresponding references, please inform the authors.

An interactive, dynamic example showing Theorem 5 is also given at the above URL for tangential quadrilateral (click on the *Link to Tangential Pentagon* button in the online sketch).

Concluding remark. Asking and answering 'what if?' questions brought us to some new (at least for us) results and insights. It is

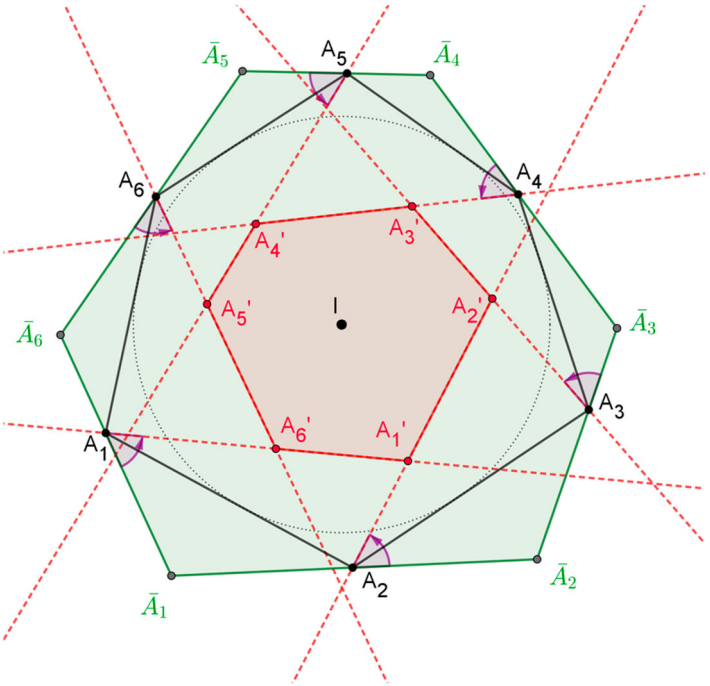

FIGURE 10. To Theorem 4, n = 6, and phi = 60°.

hoped that our examples show the generative value and potential of 'what if?' questions for the process of doing mathematics. These examples may also provide talented high school learners or undergraduate university students an opportunity for further exploration and some creative thinking.

Notes

1. We shall here deal only with the proof for the convex case, but the crossed case is very similar and left to the reader.
2. A *spiral similarity* is a special similarity transformation of geometric figures, namely a combination of a rotation by an angle about a centre and a dilation (enlargement or reduction) with the same centre. Spiral similarities produce similar figures.
3. Alternatively, these perpendiculars are called the exterior angle bisectors of the quadrilateral *ABCD*.
4. We give the proof only for the convex case, the other cases (concave, crossed) are left to the reader.

5. Thales' theorem states that, given a diameter AB of a circle, any triangle ΔABC with $C \neq A, B$ lying on the circle has a right angle at C. Its converse says that, given a triangle ΔABC with the right angle at C, the vertex C must lie on a circle with diameter AB.
6. This theorem says, roughly speaking, that angles in a circle subtending the same arc are equal.
7. Again, we give the proof only for the convex case, the other cases (concave, crossed) are left to the reader.

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