CHAPTER 3

Collinearity and Concurrence

But he opened out the hinges,
Pushed and pulled the joints and hinges,
Till it looked all squares and oblongs
Like a complicated figure
In the Second Book of Euclid.

C. L. Dodgson

After discussing some further properties of triangles and quadrangles (or quadrilaterals), we shall approach the domain of projective geometry (and even trespass a bit). A systematic development of that fascinating subject must be left for another book, but four of its most basic theorems are justifiably mentioned here because they can be proved by the methods of Euclid; in fact, three of the four are so old that no other methods were available at the time of their discovery. All these theorems deal either with *collinearity* (certain sets of points lying on a line) or *concurrence* (certain sets of lines passing through a point). The spirit of projective geometry begins to emerge as soon as we notice that, for many purposes, parallel lines behave like concurrent lines.

3.1 Quadrangles; Varignon's theorem

A *polygon* may be defined as consisting of a number of points (called *vertices*) and an equal number of line segments (called *sides*), namely a cyclically ordered set of points in a plane, with no three successive points collinear, together with the line segments joining consecutive pairs of the points. In other words, a polygon is a closed broken line lying in a plane. A polygon having n vertices and n sides is called an n-gon (meaning literally "n-angle"). Thus we have a *pentagon* ($n = 5$), a *hexagon* ($n = 6$), and so on. In fact, the Greek name for the number n is used except when $n = 3$ or 4. In these two simple cases it is customary to use the Latin forms *triangle* and *quadrangle* rather than "trigon" and "tetragon" (although "trigon" survives in the word "trigonometry"). Obviously we should discourage the tendency to call a quadrangle a "quadrilateral". (In projective geometry, where the sides are whole lines instead of mere segments, we need both terms with distinct meanings.)

Two sides of a quadrangle are said to be *adjacent* or *opposite* according as they do or do not have a vertex in common. Similarly, two vertices are adjacent or opposite according as they do or do not belong to one side. The lines joining pairs of opposite vertices are called *diagonals.* Thus a quadrangle $ABCD$ has sides AB, BC, CD, DA, diagonals AC and BD.

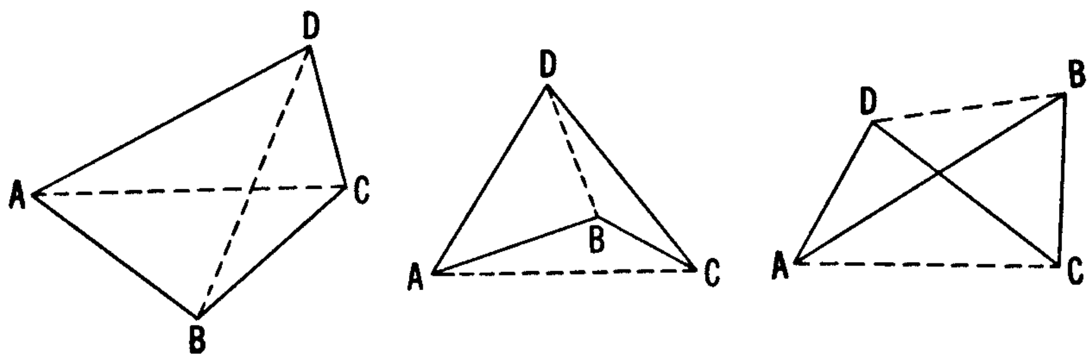

Figure 3.1A

In Figure 3.1A we see quadrangles of three obviously distinct types: a *convex* quadrangle whose diagonals are both inside, a *re-entrant* quadrangle having one diagonal inside and one outside, and a *crossed* quadrangle whose diagonals are both outside.

We naturally define the *area* of a convex quadrangle to be the sum of the areas of the two triangles into which it is decomposed by a diagonal:

$$(ABCD) = (ABC) + (CDA) = (BCD) + (DAB).$$

In order to make this formula work for a re-entrant quadrangle, we regard the area of a triangle as being *positive* or *negative* according as its vertices are named in counterclockwise or clockwise order. Thus

$$(ABC) = (BCA) = (CAB) = -(CBA).$$

For instance, the re-entrant quadrangle in the middle of Figure 3.1A has area

$$(ABCD) = (BCD) + (DAB) = (CDA) - (CBA)$$
$$= (CDA) + (ABC).$$

Finally, the formula forces us to regard the area of a crossed quadrangle as the *difference* between the areas of the two small triangles of which it is apparently composed.

When combined with the idea of directed segments (Section 2.1), the convention $(ABC) = -(CBA)$ enables us to extend our proof of Ceva's theorem and its converse (1.21 and 1.22) to cases where X or Y or Z divides the appropriate side of $\triangle ABC$ in a negative ratio, i.e., externally.

The following theorem is so simple that one is surprised to find its date of publication to be as late as 1731. It is due to Pierre Varignon (1654–1722).

THEOREM 3.11. *The figure formed when the midpoints of the sides of a quadrangle are joined in order is a parallelogram, and its area is half that of the quadrangle.*

We recall that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long as that third side. Given a quadrangle $ABCD$, let the midpoints of its sides AB, BC, CD, DA be P, Q, R, S, as in Figure 3.1B. Considering the triangles ABD and CBD, we infer that PS and QR are both parallel to the diagonal BD and equal to $\frac{1}{2}BD$. Hence the quadrangle $PQRS$ is a parallelogram;† it is often referred to as the *Varignon parallelogram* of quadrangle $ABCD$.

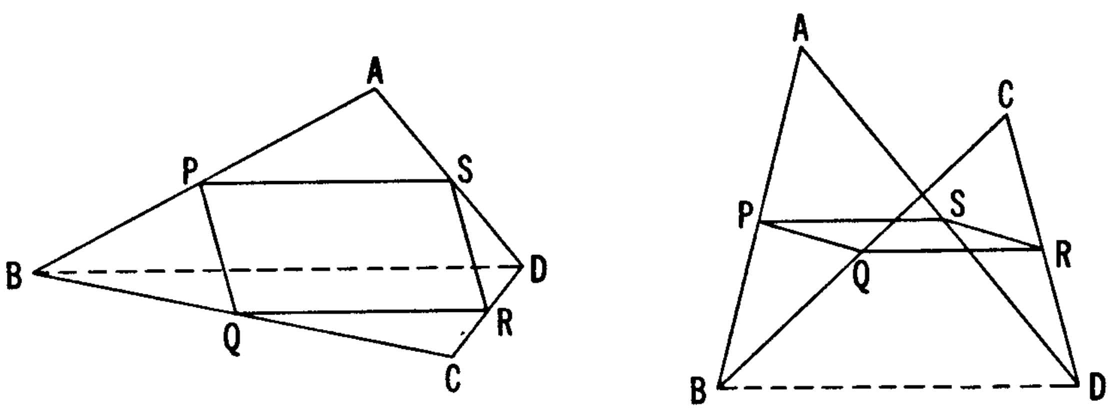

Figure 3.1B

As for the area, we have

$$\begin{aligned}(PQRS) &= (ABCD) - (PBQ) - (RDS) - (QCR) - (SAP)\\ &= (ABCD) - \tfrac{1}{4}(ABC) - \tfrac{1}{4}(CDA) - \tfrac{1}{4}(BCD) - \tfrac{1}{4}(DAB)\\ &= (ABCD) - \tfrac{1}{4}(ABCD) - \tfrac{1}{4}(ABCD)\\ &= \tfrac{1}{2}(ABCD).\end{aligned}$$

The reader may like to draw a re-entrant quadrangle $ABCD$ and verify that this decomposition is valid also in that case.

† It would still be a parallelogram if $ABCD$ were a *skew* quadrangle (not all in one plane).

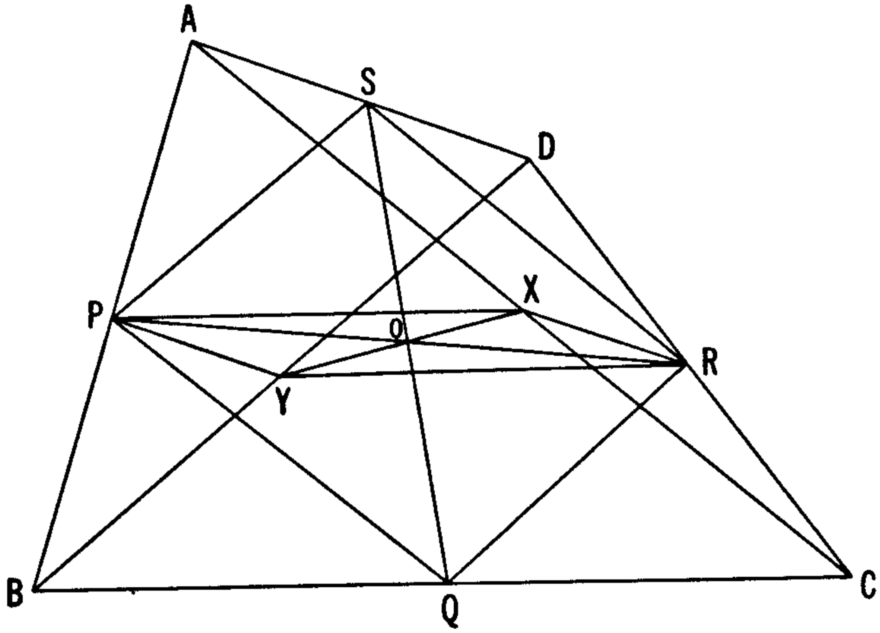

Figure 3.1C

Since the diagonals of any parallelogram bisect each other, the midpoints of PR and QS coincide at the center of the Varignon parallelogram (i.e., at that point O of Figure 3.1C). Now, just as AC and BD are the diagonals of $ABCD$, so AD and BC are the diagonals of $ABDC$. Since PR has only one midpoint, the Varignon parallelogram $PYRX$ of the new quadrangle $ABDC$ has the same center O. Hence

THEOREM 3.12. *The segments joining the midpoints of pairs of opposite sides of a quadrangle and the segment joining the midpoints of the diagonals are concurrent and bisect one another.*

(This is the first of our theorems about concurrence.)

The following result will be found useful:

THEOREM 3.13. *If one diagonal divides a quadrangle into two triangles of equal area, it bisects the other diagonal. Conversely, if one diagonal bisects the other, it bisects the area of the quadrangle.*

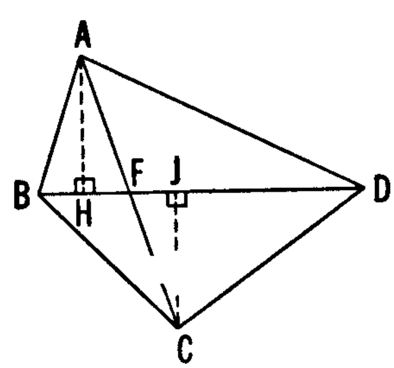

Figure 3.1D