$$\frac{CX}{b} = \frac{CX}{KJ} = \frac{BC}{BK} = \frac{a}{a+b}, \qquad \frac{CY}{a} = \frac{CY}{GI} = \frac{CA}{GA} = \frac{b}{a+b},$$

whence

$$CX = \frac{ab}{a+b} = CY.$$

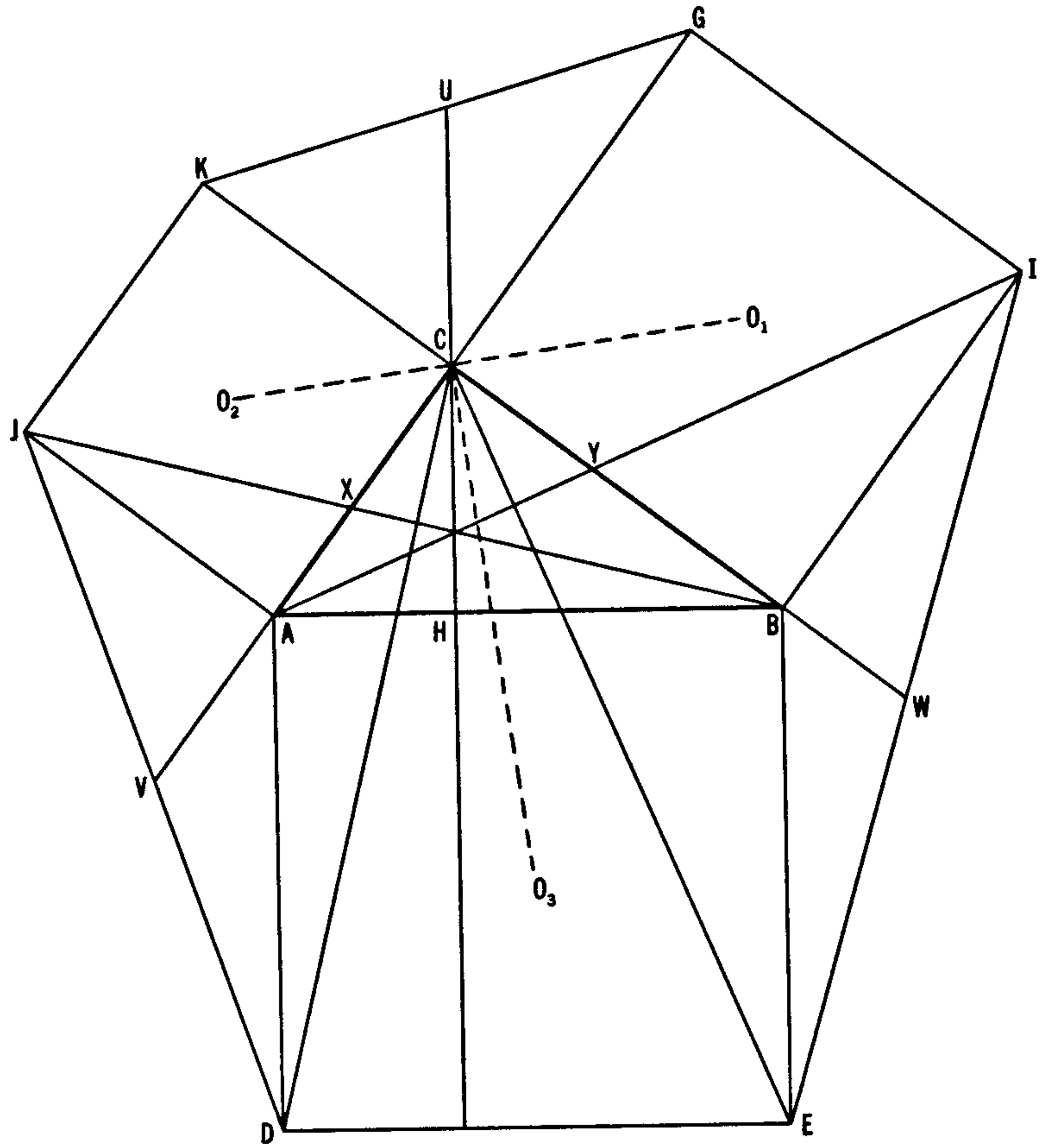

Figure 4.2B

EXERCISES

1. If squares are erected externally on the sides of a parallelogram, their centers are the vertices of a square. [29, pp. 96–97.]

2. In Figure 4.2B, (i) the three lines AI, BJ, CH are concurrent;

(ii) $O_1O_2 = CO_3$, and these lines are perpendicular; (iii) U, V, W are the midpoints of GK, JD, EI.

3. Construct an equilateral triangle such that a given point inside it is distant 2 units from one vertex, 3 units from a second vertex, and 4 units from the third vertex.

4.3 Half-turn

One kind of rotation shares with translations the property of transforming every line into a parallel line. This is the *half-turn* or rotation through 180°, which transforms each ray into an oppositely directed ray. Clearly, a half-turn is completely determined by its center. Since a translation transforms each ray into a parallel ray, the effect of two half-turns applied successively is the same as the effect of a translation: in brief, the "sum" of two half-turns is a translation (which reduces to the identity if the two half-turns have the same center). More precisely, if points A, B, C are evenly spaced along a line, so that B is the midpoint of AC, the half-turn about A leaves A invariant, and the half-turn about B takes A into C; thus the sum of these two half-turns is the translation $\overrightarrow{AC}$, and is the same as the sum of the half-turns about B and C.

Figure 4.3A illustrates the sum of half-turns about O_1 and O_2. The line segment AB is transformed first into $A'B'$ (oppositely directed) and then into $A''B''$; thus the sum is the translation $\overrightarrow{AA''} = \overrightarrow{BB''}$.

Many old and familiar theorems can be proved simply when half-turns are used. In Figure 4.3B, O is the common midpoint of two segments AC and BD. The half-turn about O, taking AB into CD, shows that $ABCD$ is a parallelogram. Again, in Figure 4.3C, M and N being the midpoints of AB and AC, we see that the sum of half-turns about these two points is the translation $\overrightarrow{MM''} = \overrightarrow{BC}$, whence MN is parallel to BC and half as long.

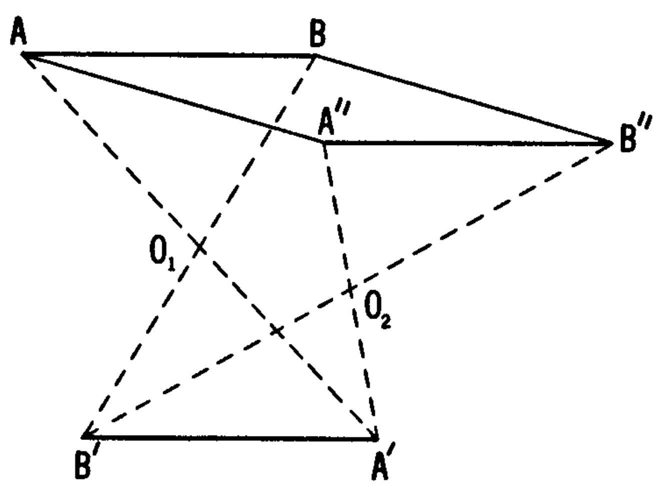

Figure 4.3A

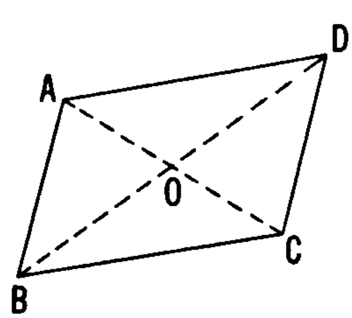

Figure 4.3B

3. Use the degenerate hexagon $AZBXCY$.

Section 4.1

1. Regarding the segment a in two ways as a vector, translate ΔABC to $\Delta A'B'C'$ on the right and to $\Delta A''B''C''$ on the left. Join the points $AB \cdot A''C''$ and $AC \cdot A'B'$.

2. A tessellation of equilateral triangles, six surrounding each vertex.

Section 4.2

1. Use quarter-turns about the centers of the squares.

2. (i) Since $CX/b = a/(a + b)$,

$$\frac{CX}{XA} = \frac{CX}{b - CX} = \frac{a}{a + b - a} = \frac{a}{b}.$$

Similarly, $BY/YC = a/b$. Also

$$\frac{AH}{HB} = \frac{(CAH)}{(CHB)} = \frac{b^2}{a^2}.$$

Since now

$$\frac{BY}{YC}\frac{CX}{XA}\frac{AH}{HB} = \frac{a}{b}\frac{a}{b}\frac{b^2}{a^2} = 1,$$

the result follows by Ceva.

(ii) ΔABC is one half of a parallelogram $ABFC$ whose center M is the midpoint of BC. Applying Exercise 1 to this parallelogram, we see that $MO_2 = MO_3$ and these lines are perpendicular. Also $MO_1 = MC$ and *these* lines are perpendicular. Hence a quarter-turn about M takes ΔMO_1O_2 to ΔMCO_3.

(iii) Complete the rectangle $KCGC'$ and the parallelograms $DAJA'$, $IBEB'$. Positive and negative quarter-turns about O_1, O_2, O_3 show that the six triangles $B'IB$, $C'CG$, $CC'K$, $JA'A$, DAA', BEB' are directly congruent to ΔABC. Hence the points U, V, W are the centers of the rectangle and parallelograms.

3. Consider the effect of a rotation through $60°$ about one vertex of the desired equilateral triangle.