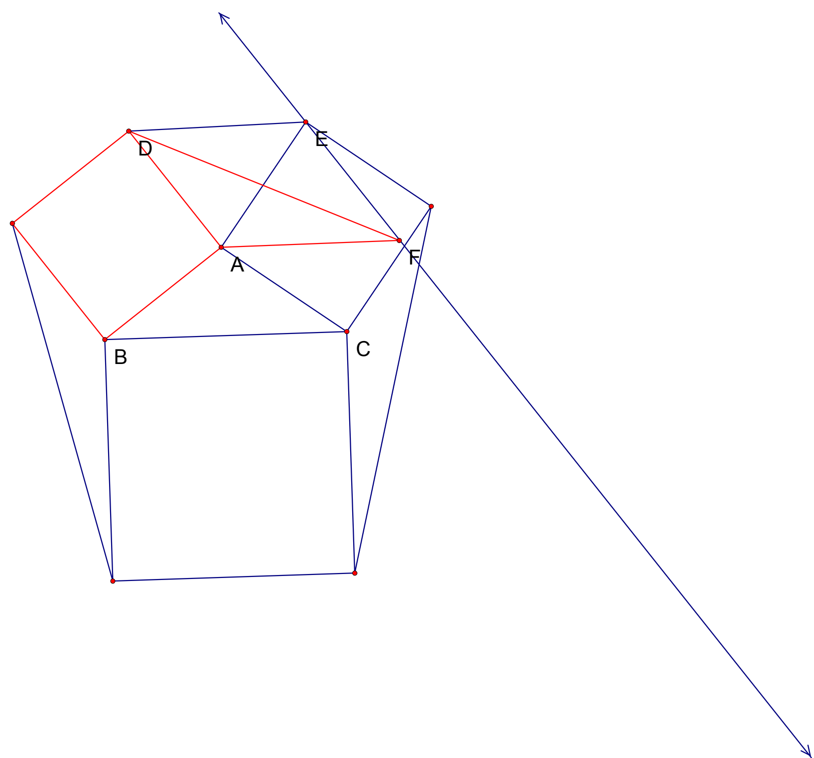
Association of Teachers of Mathematics

More info: http:/www.atm.org.uk

Responses to an article by Geoff Faux in Mathematics Teaching MT189: **Happy 21st Birthday Cockcroft**

To Geoff Faux,
Re. *Happy 21st birthday Cockcroft*, *MT189*, December 2004



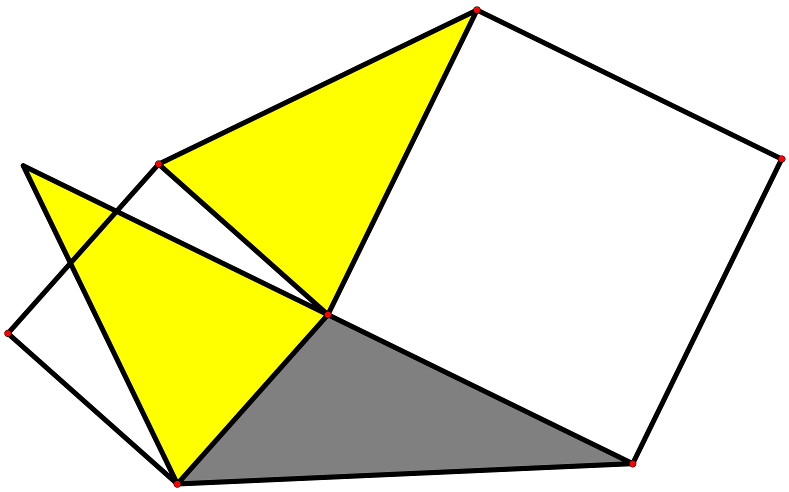
Construct a line parallel to AD through vertex of square E.
Construct point F on this line so that AE=AF
Area of triangle ADE = area of triangle ADF (same height)
Angle DAE = 180 – angle ABC (angles round point A)
AEF = DAE (alternate) = EFA (isosceles AEF)
so angle EAF = 180 – 2(180 – ABC) = ABC – 180
DAF = DAE + EAF = ABC

so triangle ABC is congruent to triangle ADF (SAS) and therefore area of ADE = area of ABC.

The arguments for other triangles would be similar.

Chris Harmer

Here is *a* response to your request about proving.



Rotate the yellow triangle through 90° about its common vertex with the grey triangle. The resulting big (yellow and grey) triangle is divided in two by its median and hence the two pieces are equal in area.

By the way, when the diagonals of a parallelogram are drawn the parallelogram is divided into four triangles of equal area – two congruent pairs. The grey triangle is such a quarter of three different parallelograms. The 'other' triangles in these three parallelograms are the three triangles which are to be proved equal in area to the grey triangle.

Derek Ball

A GEOMETRIC PROOF OF CROSS'S THEOREM

JULIAN GILBEY

In MT189 [1], Geoff Faux challenges us to find a different proof of Cross's Theorem, which states that in the geometric configuration below, where squares are drawn on the three sides of a general triangle, the four triangles all have equal areas.

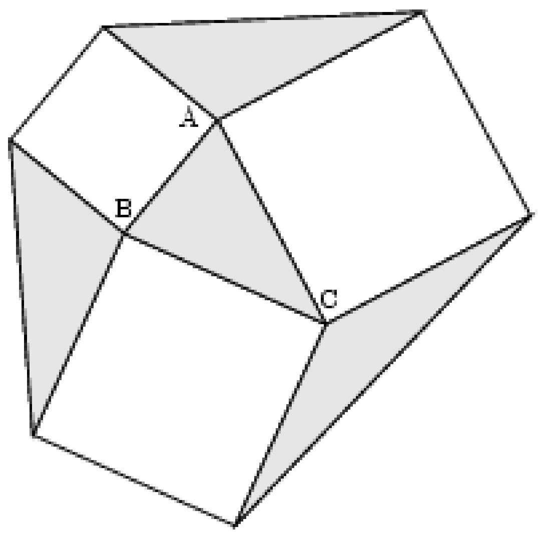


Figure 1. Equal triangles?

Our PGCE group considered this question, looking for proofs of Cross's Theorem. In order to explore the properties of this configuration, we experimented (and even played) with Cabri and Geometer's Sketchpad in order to get a feeling for the situation. (Geometer's Sketchpad allows for measuring areas, which was very useful for our investigations.) Along the way, the sketches suggested some interesting plausible conjectures; however, one of the superb features of using dynamic geometry software is that it is easy to adjust the original triangle to test such conjectures empirically, and some of the conjectures were thus quickly consigned to the "failed attempts" basket.

We succeeded in producing the following proof. Consider figure 2. We aim to show that the two shaded triangles, ABC and PBQ have the same area; the result then follows immediately. We have drawn in extensions of the segments AB and PB, and constructed parallels to them through P and Q respectively. We label the intersection of AB extended with the parallel through Q as X. We also dropped a perpendicular to PB extended from vertex C, meeting PB extended at Y. Now, the area of PBQ, half base times height, is given by $\frac{1}{2}PB.BX$, and the area of ABC is $\frac{1}{2}AB.BY$. (Note that BY is perpendicular to AB, since $\angle PBA$ is a right angle.) But $BC = BQ$ (sides of a square); $\angle BXQ$ and $\angle BYC$ are both right angles, and $\angle XBQ$ and $\angle YBC$ are easily seen to be equal. Thus BXQ and BYC are congruent, so $BX = BY$. Since $PB = AB$, it follows that the two triangles PBQ and ABC have equal area.

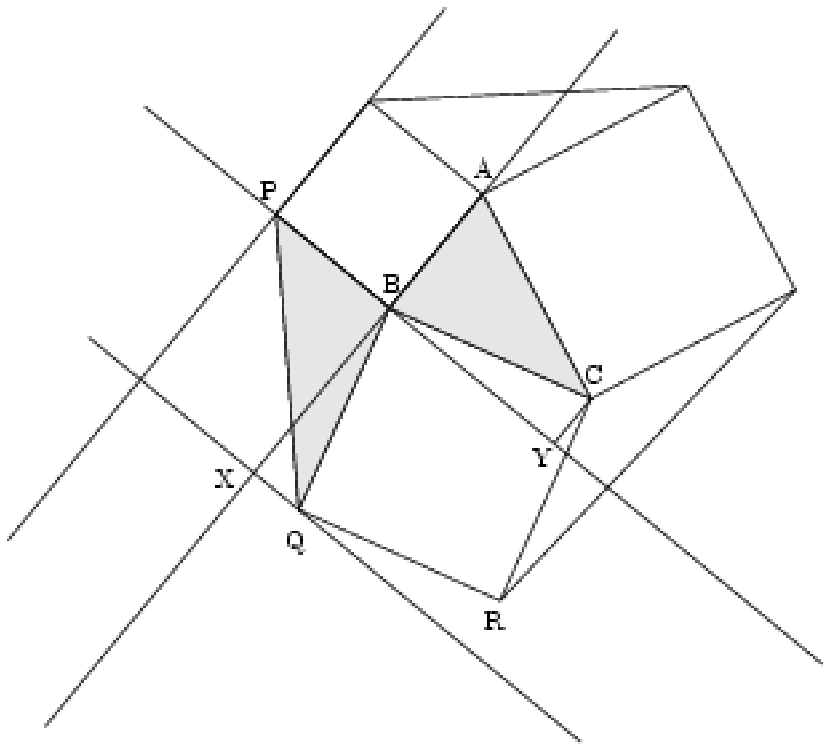


FIGURE 2. A geometric proof of Cross's Theorem

Although this argument might seem to rely on the angle ABC being acute, it is important to check that the results do not depend upon the actual configuration. (This is often missed in textbooks proving the result that the angle at the centre of a circle is twice that at the circumference: they only consider one possible configuration.) A quick look at figure 3, however, shows that the argument follows identically if the angle is obtuse. The situation is trivial if ABC is a right angle.

Paul Andrews, our mathematics lecturer, then suggested extending the result to quadrilaterals, pentagons and even possibly beyond. Our investigations then led to an interesting result for quadrilaterals, but failed to reveal anything for pentagons or beyond.

The benefits of using dynamic geometry software for visualising geometric scenarios became very clear to us. Their incredible ability for also showing which features of a problem are essential and which are coincidental also helped us to comprehend the essence of the situation.

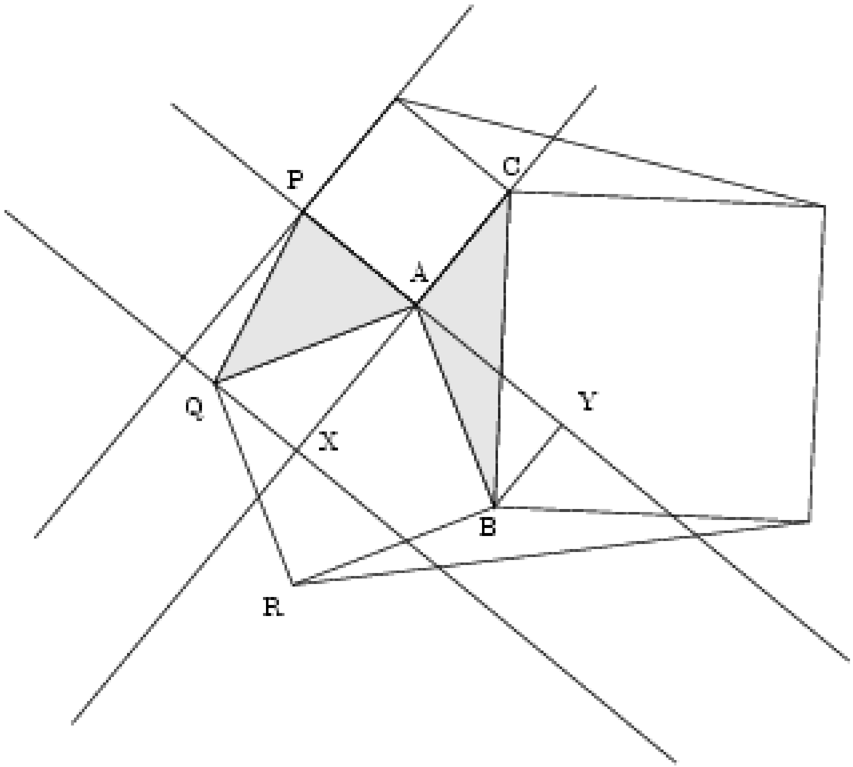


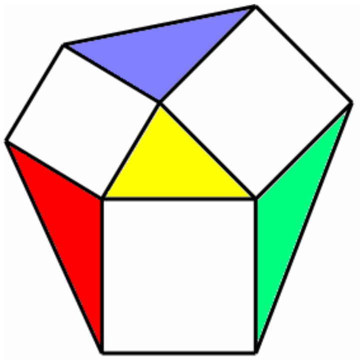
Figure 3. A different configuration

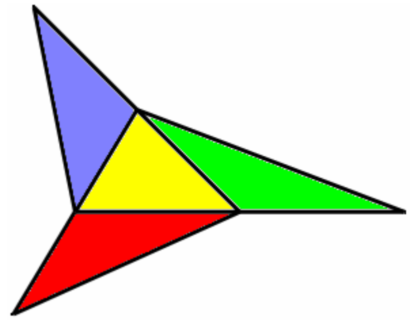
Dr Julian Gilbey is currently a PGCE student at the Faculty of Education, University of Cembridge.

References

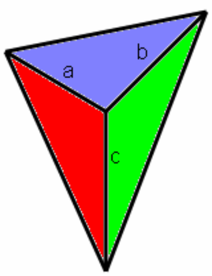
[1] Geoff Faux: Happy 21st Birthday Cockcroft, **MT189**, December 2004, p10.

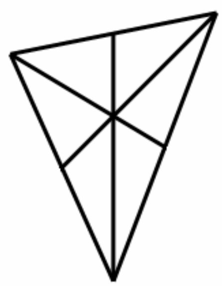
Derek's Response to Geoff – and then a bit further.



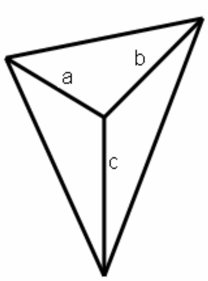


With reference to Geoff Faux's challenge in MT189, p10, and Derek Ball's response in MT190 p16, to start with, I did the same as Derek and got the cheerful little shape above.
I then thought that I could move the three triangles in a different way. So I moved them towards each other, successfully eliminating the three squares and the central triangle. The three triangles have the same area, that we know.
Aha!! If I extend the lengths a, b and c I have drawn the three medians of the triangle which meet at the centre of gravity.





So, let's start at the end and go backwards!!
Start with any triangle and draw the medians as far as their point of intersection. I have divided the triangle into three triangles of equal area.



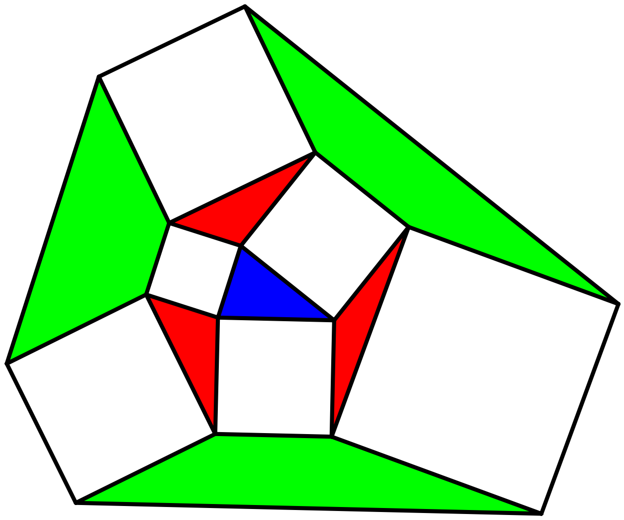
If I now construct a triangle with sides a, b and c this triangle has the same area as these three triangles, i.e. an area one third of the original triangle.
WOW!!!!! I never knew that.

David Cain

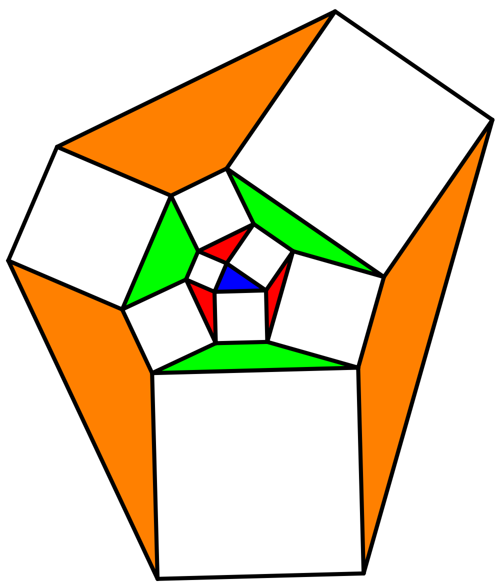
Working on Cross's theorem

Derek Ball

At about the time MT190 appeared, Geoff Faux pointed out to me in a telephone conversation that if you take the three outer triangles in the Cross's theorem diagram and draw squares on them, then the three quadrilaterals you get in between all have the same area, which is five times the area of the original triangle.



I printed this picture and shared it with Barbara, who suggested that the four quadrilaterals looked like trapezia. While she was trying to prove that, I had added squares on the free edges of the quadrilaterals and produced some more quadrilaterals which were, according to GSP, all 24 times the area of the original triangle.



1, 5, 24 …   Hmm!

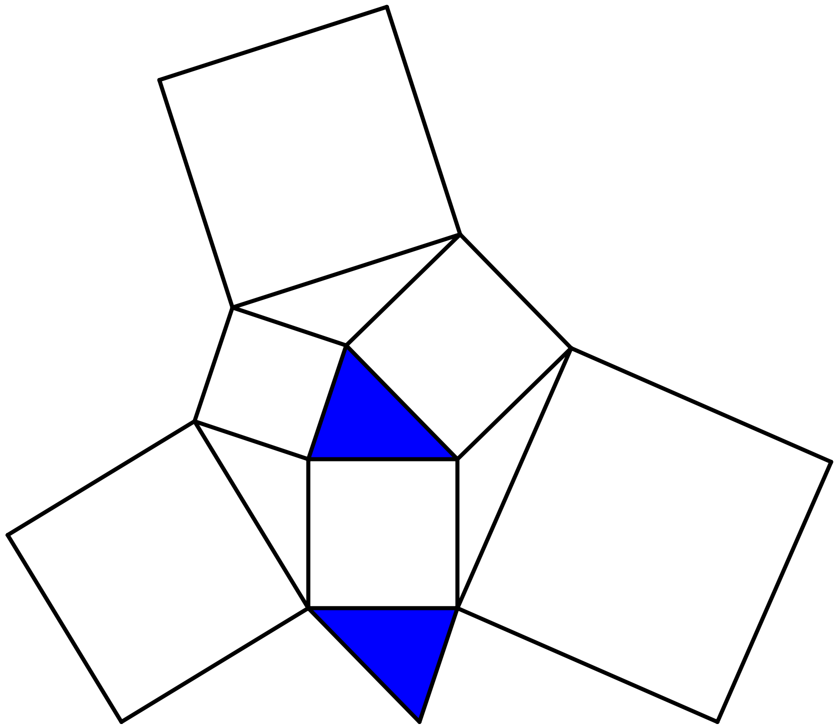
Barbara was downstairs doing some angle chasing to establish her trapezium property. We had a discussion about this before I raced back upstairs to GSP and discovered that the next quadrilaterals – or trapezia – had an area of 115 apparently. Why 115?

I had some wild idea that these numbers were going up roughly by a factor of 5 each time and so with a bit of adjustment I obtained this:

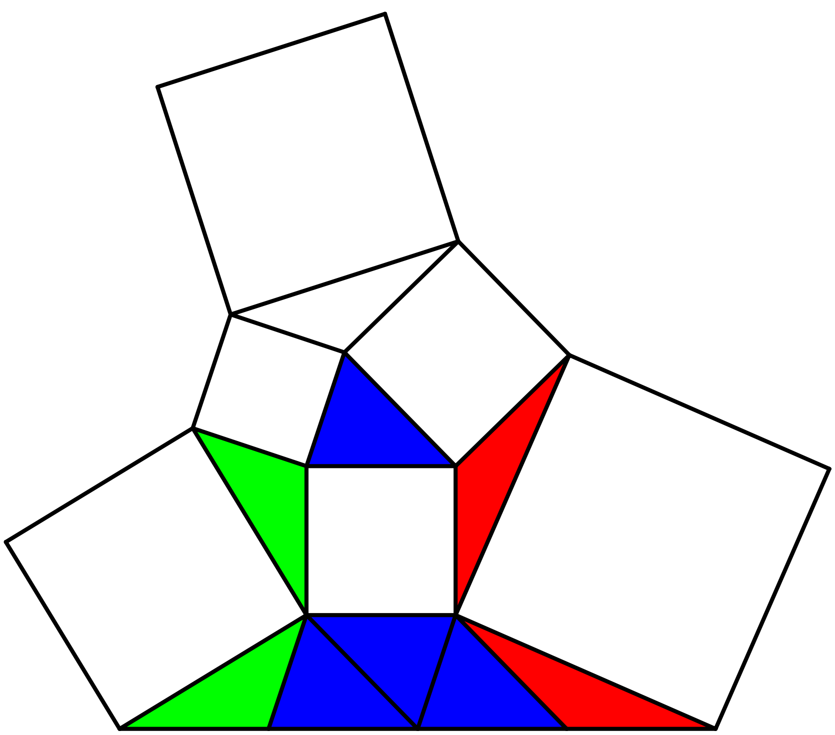
$5 \times 5 - 1 = 24$
$5 \times 24 - 5 = 115$

This led me to predict that the next quadrilaterals would have an area of 551. Having reduced the size of my picture sufficiently to add the next set of squares and quadrilaterals I obtained … 551. Wow!

Downstairs Barbara had decided that a good move was to rotate the centre triangle through half a turn and put it inside one of the quadrilaterals. She thought it would fit.

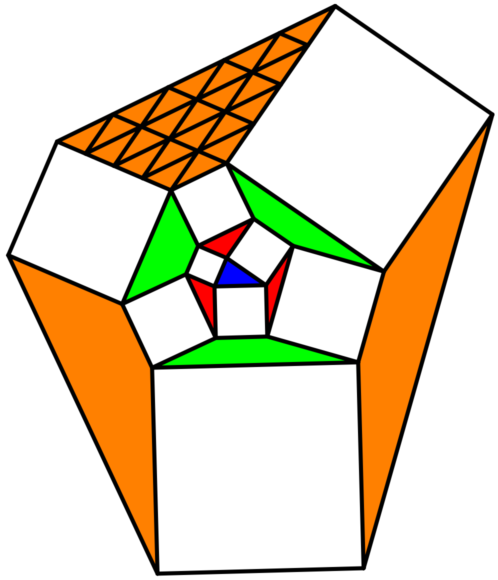


I suggested translating the same triangle to go either side of this new triangle.



With a bit of angle chasing and use of congruent triangles we were able to prove that these triangles did indeed fit, that the triangles created either side were congruent to two of the ‘Cross’ triangles, as indicated by the coloured picture and that the quadrilateral was indeed a trapezium with area five times that of the original triangle.

After this insight, proving the 24 followed relatively quickly.

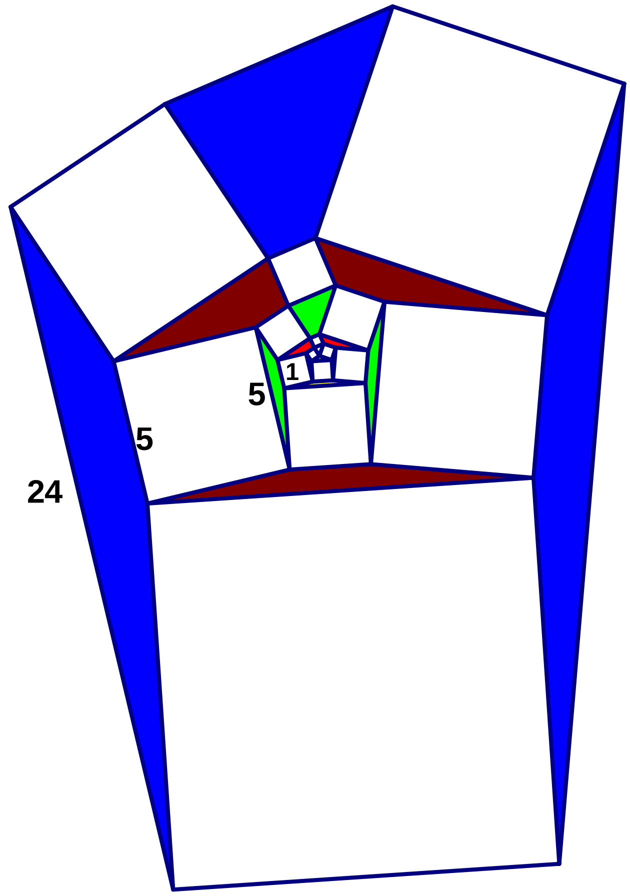


We were now getting the hang of things quite well. In deciding how we could generalise and hence prove that the areas are given by the sequence mentioned already,

1, 5, 24, 115, 551, 2640, 16249 …

we started looking at the ratios of lengths of parallel sides of trapezia. We discovered to our amazement that one set of lengths of the sides began

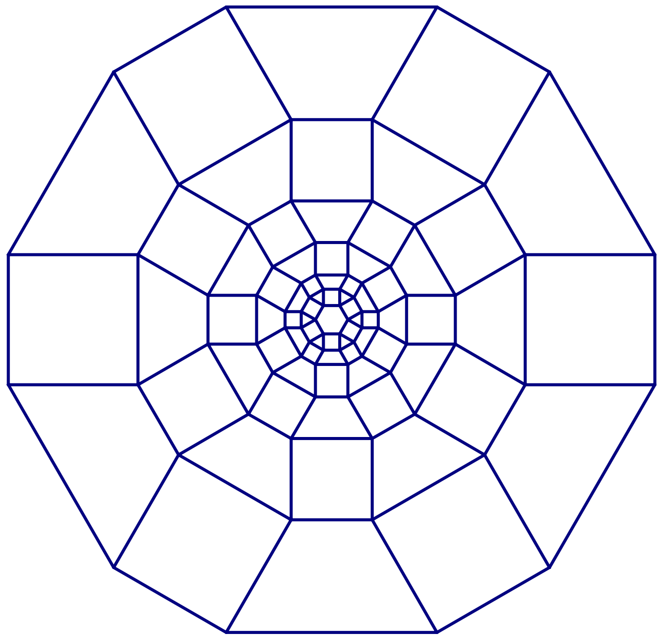
1, 5, 24, ..



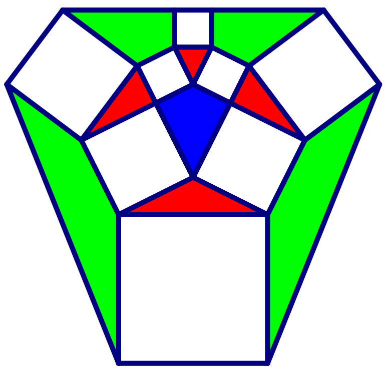
Why was the pattern of lengths of the lengths of lines identical to the pattern of areas of trapezia, when we were calculating these two sequences in quite different ways. Progress was made on this at BCME6 when I had a conversation with Toni Beardon about proving an identity involving squares of consecutive terms of the Fibonacci sequence. I used her method of proof to prove a similar result about terms of the 1, 5, 24, … sequence.

Our thoughts turned to further generalisations. Starting with a square in the middle gives a pattern that is easy to explore on squared paper. The same patterns of areas and lengths are preserved if the shape in the middle is a parallelogram instead of a square.

Starting with a regular hexagon in the middle gives a different way to generate the Fibonacci sequence. I shall leave you to work out the details for yourself.



Then we started fiddling about with the general quadrilateral. There did not seem to be any tidy patterns in the areas. Then Barbara started drawing right-angled kites on squared paper and adding up the areas of the shapes obtained for each round.



A pattern appeared! Would the pattern work for quadrilaterals? And how could it be proved?

For more thoughts about all this and some Geometer's Sketchpad files go to the ATM website.