Dirk Laurie's Hexagon Problem

Michael N. Fried

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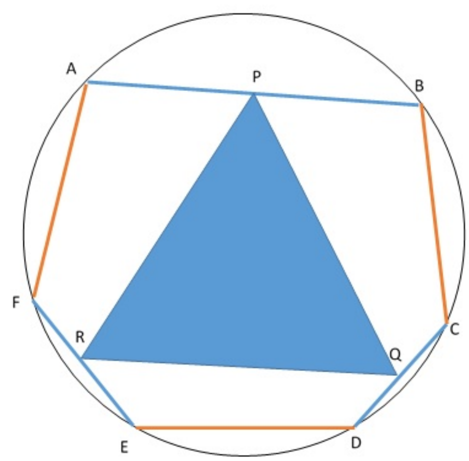
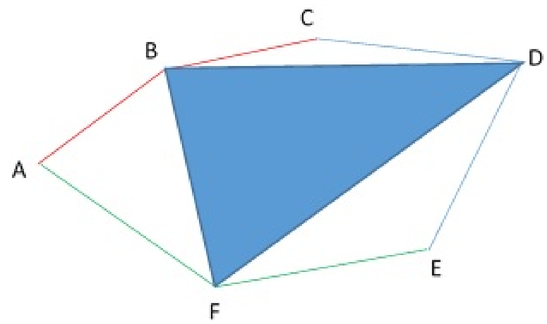
The following problem was posted by Michael de Villiers, who said, in turn, that it was given to him by a colleague, Dirk Laurie.

Laurie's Problem: *Suppose $ABCDEF$ is a hexagon inscribed in a circle with unit radius, $AF = BC = ED = 1$, and P, Q, R are the midpoints of the sides AB, CD, EF, respectively. Show that PQR is an equilateral triangle.*



One solution suggested by Michael de Villiers uses another interesting theorem proven by Waldemar Pompe of Warsaw University. It states:

> **Pompe's Theorem**: *Let $ABCDEF$ be a hexagon such that $AB = BC, CD = DE, EF = FA$ and the sum of the angles B, D, E is 2π (thus also the sum of the angles A, C, E is also 2π). Then the angle $BDF = \frac{1}{2}EDC$, $FBD = \frac{1}{2}ABC$,$DFB = \frac{1}{2}EFA$.*

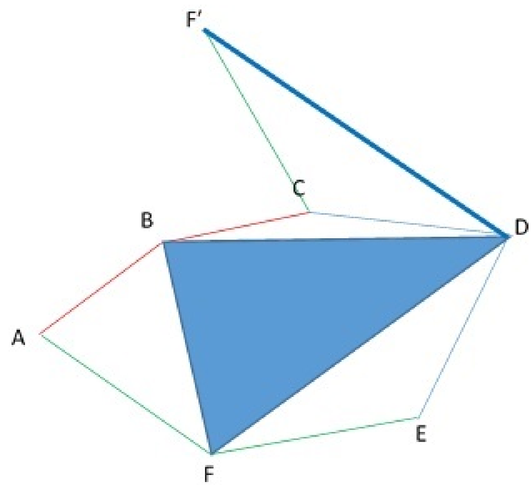


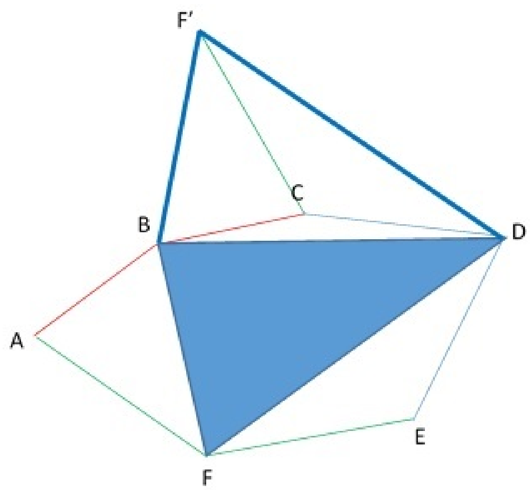
The theorem is easy to prove.

Let the triangle DEF be rotated through angle EDC. Since $ED = DC$, E will be moved to C and F to a point F' (see the figure below). Moreover, the angle $F'DF$ is equal to DEF and, of course, $DF = DF'$

Note that since the sum of the angles A, C, E is 2π and $BCD = C$ and $F'CD = E$, we have $BCF' = A$. Also, $F'C = AF$ and $AB = BC$. Therefore, the triangles $BF'C$ and BFA are congruent, so that $BF' = BF$. From the congruence of the triangles, we see too that $BF'C$ could have been obtained from BFA via a rotation about B through angle ABC; that means that angle $ABC = F'BF$.

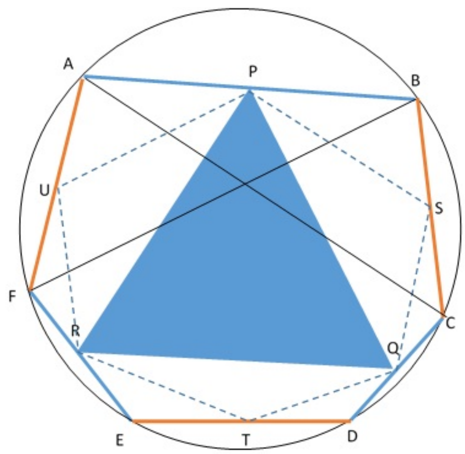
Since $BF' = BF$ and $DF = DF'$, $BF'DF$ is a "dart," from which we conclude that BD bisects angles $F'BF$ and $F'DF$, so that $BDF = \frac{1}{2}F'DF = \frac{1}{2}EDC$ and $FBD = \frac{1}{2}F'BF = \frac{1}{2}ABC$. Similarly we can show that $DFB = \frac{1}{2}EFA$.





So, how does Michael de Villiers use this theorem to solve Laurie's problem?

Let U, S, T be the midpoints of of AF, BC, ED respectively. Complete the hexagon, $PSQTRU$. Join FB and AC: these are equal since both subtend an arc equal to $AB + \frac{\pi}{3}$. Moreover, $UP||FB$ and $PS||AC$ and $UP = PS = \frac{1}{2}FB$. Also since arc $AF = \frac{\pi}{3}$, angle $APU = ABF = \frac{\pi}{6}$. Similarly, $BPS = \frac{\pi}{6}$, so that $UPS = \frac{2\pi}{3}$. By the same argument, we have $URT = TQS = \frac{2\pi}{3}$ and also, $UR = RT, TQ = QS$. Therefore, the hexagon $PSQTRU$ satisfies the conditions of Pompe's theorem, and, therefore, the angles $RPQ = PQR = QRP = \frac{1}{2}\frac{2\pi}{3} = \frac{\pi}{3}$.



Here are two other solutions to the problem.

The first is via complex numbers. Let $\lambda = e^{i\frac{\pi}{3}}$ so that $\lambda^2 = \lambda - 1$ and $\lambda^3 = -1$, and let the rotations through arcs AB and CD be β and δ, respectively. Then we have:

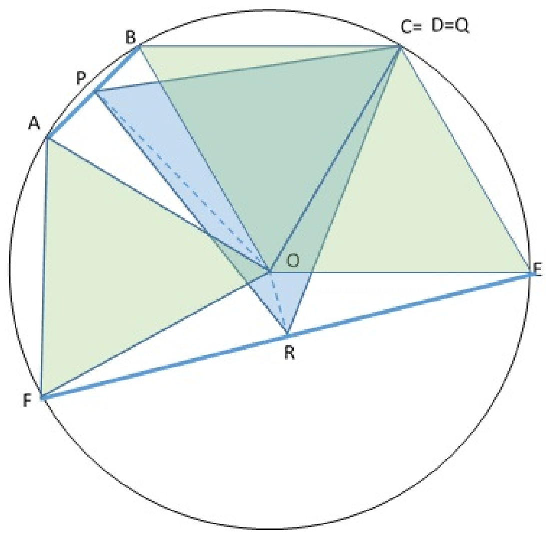
$$A = \lambda F$$
$$B = \lambda\beta F$$
$$C = \lambda^2\beta F$$
$$D = \lambda^2\beta\delta F$$
$$E = -\beta\delta F$$

Also, we have:

$$P = \frac{1}{2}(A + B)$$
$$Q = \frac{1}{2}(C + D)$$
$$R = \frac{1}{2}(E + F)$$

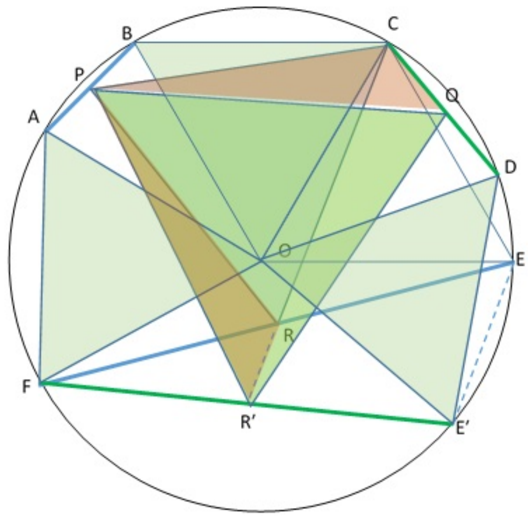
Thus, we have to show that $P - Q = \lambda(R - Q)$ or $2(P - Q) = 2\lambda(R - Q)$.
Now, $2(P-Q) = A+B-C-D = (\lambda+\lambda\beta-\lambda^2\beta-\lambda^2\beta\delta)F = (\lambda+\beta-\lambda^2\beta\delta)F$
And $2(R-Q) = E+F-C-D = (-\beta\delta+1-\lambda^2\beta-\lambda^2\beta\delta)F = (1-\lambda^2\beta-\lambda\beta\delta)F$
But $(\lambda + \beta - \lambda^2\beta\delta) = \lambda((1 - \lambda^2\beta - \lambda\beta\delta)$, so we're done.

A different geometric proof is the following. It was my original proof of the proposition after Michael de Villiers posted it. In my way of visualizing the problem, I consider the chords AF, BD, ED as sides of three equilateral triangles hinged at the center of the circle. Initially, these triangles are side by side, that is, $A = B$ and $C = D$. In this initial state, P also coincides with A and B, Q with C and D, and R with the center of the circle, O, so that the triangle PQR coincides with the equilateral triangle whose side is chord BC. We will now proceed in two steps to rotate the triangles into a general position. First, rotate the leftmost triangle AOF counterclockwise to an arbitrary position as in the figure below.



Again, P and R are the midpoints of AB and EF, respectively, and Q, the midpoint of CD, still coincides with C and D. Join OR amd OP. Since angle $FOE = \pi - AOB$, we have $ROE = \frac{\pi}{2} - POB = PBO$. Since triangles PBO and ROE are right triangles and $OB = OE$, they are congruent. Hence, $PB = OR$. Since angle $PBC = \frac{\pi}{3} + PBO = \frac{\pi}{3} + ROE = ROQ$, we can rotate triangle PBQ into ROQ about Q through an angle of $\frac{\pi}{3}$. Thus the line PQ is rotated into the line RQ also through an angle of $\frac{\pi}{3}$, from which it follows immediately that PQR is equilateral being isosceles with an apex angle equal to $\frac{\pi}{3}$.

Now, from this position we will rotate triangle COE clockwise to an arbitrary position DOE' as in the next figure. In order to leave traces of the previous position, we are using E' and R' for the third vertex of the rotating equilateral triangle and midpoint of FE'.



So, consider. From before, $PC = PR$ and $RPC = \frac{\pi}{3}$. Also since by the rotation of the equilateral triangle COE to position DOE' we can see that $CD = EE'$ and it has been rotated through an angle of $\frac{\pi}{3}$.

Moreover, since R and R' are the midpoints of FE and FE' respectively, $RR' = \frac{1}{2}EE' = CQ$ and since RR' is also parallel to EE', we can see that CQ has also been rotated through an angle of $\frac{\pi}{3}$ to position RR'.

Hence the triangle PCQ has been rotated through an angle of $\frac{\pi}{3}$ about P to the position of triangle PRR' (I have shaded that triangle) so that also PQ is rotated through $\frac{\pi}{3}$ to PR'. Therefore, by the same reasoning as above, triangle PQR' is equilateral.