The Fermat-Steiner Problem
Author(s): Shay Gueron and Ran Tessler
Source: *The American Mathematical Monthly*, Vol. 109, No. 5 (May, 2002), pp. 443-451
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2695644
Accessed: 09/09/2008 14:59

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/action/showPublisher?publisherCode=maa.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit organization founded in 1995 to build trusted digital archives for scholarship. We work with the scholarly community to preserve their work and the materials they rely upon, and to build a common research platform that promotes the discovery and use of these resources. For more information about JSTOR, please contact support@jstor.org.

http://www.jstor.org

The Fermat-Steiner Problem

Shay Gueron and Ran Tessler

Given a triangle $\triangle ABC$, how can we find a point P for which $PA + PB + PC$ is minimal?

This problem was originally proposed by Fermat some 300 years ago, and since then has reappeared in the literature with different variations, solutions, and credits. In one reincarnation it became known as *Steiner's problem*, and we therefore call it the *Fermat-Steiner problem*. In this paper we explore the weighted Fermat-Steiner Problem, a 200-year-old generalization of the original problem.

1. THE EVOLUTION OF THE FERMAT-STEINER PROBLEM AND ITS WEIGHTED GENERALIZATION. We start by stating the problem formally. For a given a triangle $\triangle ABC$ and a point P in its plane, we define $\mathcal{G} \equiv \mathcal{G}(P) = \mathcal{G}(P, \triangle ABC) = PA + PB + PC$.

The Fermat-Steiner problem. Given a triangle $\triangle ABC$, find a point P (in its plane) such that $\mathcal{G}(P, \triangle ABC)$ is minimized.

The original treatment of the problem considered only triangles with acute angles, for which P must be an interior point. Therefore, a discussion of the relative position of P with respect to the triangle did not emerge. Let us first make some simple observations about this relative position: P cannot lie outside $\triangle ABC$ because in this case the "nearest point projection", call it P', gives $\mathcal{G}(P') < \mathcal{G}(P)$. By the convexity of the distance function it follows that if P is inside $\triangle ABC$, it is unique. Also, it is clear that if P lies on a side of $\triangle ABC$, it must be a vertex. To conclude: our minimizing candidates are the interior points and the vertices of $\triangle ABC$.

Courant and Robbins [**4**, pp. 354–359] attribute this problem to the famous 18th century Swiss geometer Jacob Steiner who studied it, probably independently of the earlier references that we are aware of today. Steiner derived a systematic solution that covers all cases [**9**, pp. 24–35]. His answer can be summarized as follows: If all of the angles of $\triangle ABC$ are less than 120°, then P is the point inside $\triangle ABC$ from which its sides are seen at the angle 120°. If one angle of $\triangle ABC$ is at least 120°, then P is the vertex at this angle.

History, however, indicates that the roots of our problem and its solutions are much earlier than Steiner's analysis. It was Fermat (1601–1665) who proposed this problem to Torricelli (1608–1647). Torricelli solved the problem and passed it along to his student Viviani (1622–1703), who published his own and Torricelli's solution in 1659. The earliest written discussion of this problem that we were able to trace is found in Cavallieri's book from 1647; see [**1**], [**13**, pp. 443–444].

Torricelli's solution uses what we call today Viviani's Theorem: *the sum of the distances of any interior point M from the sides of an equilateral triangle equals the altitude of the triangle.* To solve the problem, Torricelli considered a point P inside the triangle from which the sides AB, BC, CA are seen at the angle 120°. He constructed an auxiliary triangle whose sides pass through A, B, C and are perpendicular to PA, PB, PC, respectively. This auxiliary triangle is equilateral (see Figure 1), and applying Viviani's Theorem solves the problem.

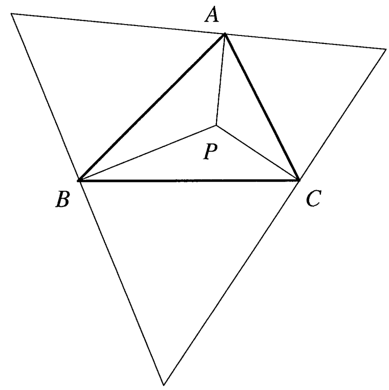

Figure 1. Torricelli's auxiliary construction.

Torricelli's solution misses some possible cases because the subtlety involved with the a priori assumption that P lies inside $\triangle ABC$ escaped his attention (he actually considered only triangles with acute angles). Unlike Steiner's motivated step-by-step derivation, it is not clear how Torricelli guessed the location of P, nor how he thought of the auxiliary construction.

In [**2**, pp. 21–23] and [**9**, pp. 24–35] we find a different approach to the Fermat-Steiner problem, which is due to Hofmann [**8**], though Honsberger [**9**, pp. 24–35] points out that the credit for this method should be shared with the Hungarian mathematician Gallai. For any point P inside $\triangle ABC$ consider the pure rotation by 60° around B, and denote the images of A and P by C' and P', respectively; see Figure 2.

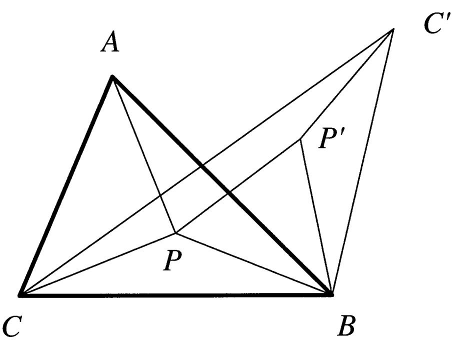

Figure 2. Hofman's solution via rotation by 60° about B.

The triangles $\triangle PBP'$ and $\triangle ABC'$ are equilateral, $AP = C'P'$, and $PB = PP'$, and therefore $AP + BP + CP = C'P' + P'P + PC$. On the other hand, the path from C to C' would be minimal if C, P, P', and C' were collinear. Therefore, to obtain a minimum, we need $\angle C'P'B = 120°$, and then $\angle APB = 120°$. Similarly, $\angle APC = \angle CPB = 120°$ at the minimizing point.

Hofmann's solution provides another characterization of the minimizing point (again, assuming in advance that the angles of $\triangle ABC$ are smaller than 120°, which implies that P is an interior point): if one erects (outwards) three equilateral triangles $\triangle BCA'$, $\triangle CAB'$, $\triangle ABC'$ on the sides of $\triangle ABC$, then the three segments AA', BB', CC' have equal lengths and pass through one point, which is the minimizing point P. This point is known today as the *Fermat point*, and also as the *isogonic center* of the triangle. Returning to history, it turns out that the relation between the isogonic center and the Fermat-Steiner problem is older than Hofmann's 1929 solution. It is mentioned (together with some interesting historical notes) in an 1897 paper by Mackay [**10**].

The Fermat-Steiner problem has several other solutions. Pedoe [**12**, pp. 11–12] proposes a solution that is based on Ptolemy's Inequality, and mentions another construction of the Fermat point, due to Sokolowsky [**12**, Exercise 1.8]. For a historical account and several solutions, see [**9**, pp. 24–35]. One solution is based on a mechanical argument that leads to the vectorial balance

$$\frac{\vec{PA}}{\|\vec{PA}\|}+\frac{\vec{PB}}{\|\vec{PB}\|}+\frac{\vec{PC}}{\|\vec{PC}\|}=0,$$

where P is an interior point, and then to the appropriate characterization of P. Solutions based on calculus are described in [**11**] and in [**14**].

The Fermat-Steiner problem inspired several generalizations. One example [**4**, pp. 359–361] is the minimization of the total length of a network emanating from *several* unknown points, and connecting n given points. In this paper we explore a generalization of the Fermat-Steiner problem that is obtained by assigning weights to the distances from P to the vertices of the triangle, namely:

The weighted Fermat-Steiner problem. Consider a triangle $\triangle A_1A_2A_3$ and three positive constants $\lambda_1, \lambda_2, \lambda_3$. Construct a point P for which the sum

$$\mathcal{F}=\mathcal{F}(P)=\mathcal{F}(P,\lambda_1,\lambda_2,\lambda_3,\triangle A_1A_2A_3)=\lambda_1PA_1+\lambda_2PA_2+\lambda_3PA_3$$

is minimal.

This weighted variant of the Fermat-Steiner problem also has old roots and several reincarnations, which we attempt to track.

The first written discussion of the weighted problem is probably due to Thomas Simpson in 1750 [**17**]. Some related investigation was presented (orally) by Nicolas Fuss to the Petersburg Academy of Sciences in 1796 [**6**]. The discussion of the weighted Fermat-Steiner problem emerged originally in a purely geometric context, and motivated presentations came afterwards.

Hugo Steinhaus, in his 1951 book *Mathematical Snapshots* (published first in Polish in 1937) introduces the weighted Fermat-Steiner problem with the following motivation [**16**, pp. 113–118]: three villages having 50, 70, and 90 pupils want to build a common school at a spot where the overall distances walked by the pupils to the school is minimized. Two approaches to the problem are mentioned briefly in the snapshot: one is a mechanical solution and the other is a solution based on a Torricelli-style construction extended to the weighted case. From the snapshot it appears that these solutions were known before Steinhaus, but no references are provided. The discussion of these two approaches is expanded in subsequent sections.

The historical track takes us from Steinhaus to Greenberg and Robetello [**7**], who rediscovered the generalized Fermat-Steiner problem in 1965, as "the three factory problem"; they proposed a rather cumbersome solution.

A special case of the weighted Fermat-Steiner problem, where the weights are the lengths of the sides of the given triangle, was published by Stensholt in 1956 in this MONTHLY [**15**]. The same result reappeared in 1983 as a problem in *Crux Mathematicorum*, with a solution in 1984 [**3**].

In 1995, Tong and Chua discussed the same problem, calling it "the generalized Fermat's point" [**19**]. They proposed a geometric solution based on extending the construction of the isogonic center to the weighted case. Instead of erecting three equilateral triangles on the sides of the given triangle, they erect (outwards) three *similar triangles* whose sides ratio is $\lambda_1 : \lambda_2 : \lambda_3$. The corresponding isogonic lines meet at one point, which is the desired minimizer P.

Another recent reference proposed an algebraic-trigonometric solution to the weighted problem [**22**].

One difference between the original Fermat-Steiner problem and its weighted variant is the analysis of the relative position of P with respect to the triangle. This discussion is more subtle in the weighted problem. A necessary condition for P to lie inside the triangle is that the weights λ_i satisfy the triangle inequality. However, this condition is not sufficient, as Steinhaus explained in his discussion of the three villages [**16**, p. 117]: "the auxiliary triangle can fail because it does not exist when one village has more children than the other two together. But even if it exists, the construction of circles on the map may fail to give an interior point of the triangle of villages. (Why?)". In the next section we resolve Steinhaus's enigmatic ending and spell out an appropriate constraint on $\triangle A_1A_2A_3$ that ensures that P is an interior point.

To continue, we agree on the following notations: a cyclic-modulo-3 notation is used for the subindex $i = 1, 2, 3$; the lengths of the sides of $\triangle A_1A_2A_3$ and the measures of its angles are denoted by $|A_{i+1}A_{i+2}| = a_i$ and $\angle A_i = \alpha_i$, respectively; the half perimeter is denoted by $p = \frac{1}{2}(a_1 + a_2 + a_3)$; the inradius is denoted by r, the circumradius by R; and the area is denoted by $S_{\triangle A_1A_2A_3} = S$.

2. A MECHANICAL SOLUTION. The approach we study in this section attacks the weighted problem from an unexpected angle. The earliest reference we know for this solution is Steinhaus's book, but this approach was probably known even earlier. Other authors in subsequent years ([**5**, pp. 361–363], [**9**, pp. 24–35], and [**18**]) used a simplified version of the same approach to solve the original Fermat-Steiner problem.

Suppose that the minimizing point P lies inside $\triangle A_1A_2A_3$, define the vectors $v_i = \overrightarrow{PA_i}$ and the unit vectors $\vec{V_i} = v_i/\|v_i\|$, and denote $\angle A_{i+1}PA_{i+2} = \phi_i$. Imagine that $\triangle A_1A_2A_3$ lies on a horizontal table, and that holes are drilled at the vertices, where smooth pulleys are attached. Three massless strings that emanate from a knot are passed through the pulleys, and three masses λ_1, λ_2, λ_3 are suspended from the ends of these strings; see Figure 3.

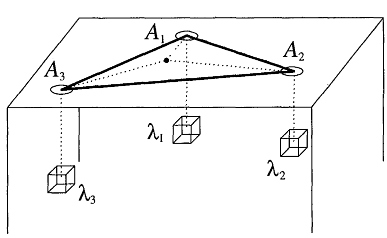

Figure 3. Solution by a mechanical argument.

Suppose that the system is released and reaches its mechanical equilibrium, and that the knot stops at an interior point of $\triangle A_1A_2A_3$. Now, apply the "minimum energy principle": at equilibrium, the sum of the distances from the floor to the three weights is minimal. This implies that the knot stops exactly at the desired minimizing point P. Since the system is in mechanical equilibrium, we have the vectorial balance

$$\lambda_1\vec{V_1} + \lambda_2\vec{V_2} + \lambda_3\vec{V_3} = 0, \tag{1}$$

or

$$\vec{V}_1 = -\frac{\lambda_2}{\lambda_1}\vec{V}_2 - \frac{\lambda_3}{\lambda_1}\vec{V}_3, \tag{2}$$

which (using the cosine law) leads to

$$\cos\phi_1 = -\frac{\lambda_2^2 + \lambda_3^2 - \lambda_1^2}{2\lambda_2\lambda_3}. \tag{3}$$

Therefore, P is the point from which A_2A_3 is seen at the angle $\phi_1 \in (0, 180°)$ defined by (3). Similarly, the sides A_1A_3 and A_1A_2 are seen from P at the angles ϕ_2 and ϕ_3, respectively, where these angles are defined by

$$\cos\phi_i = -\frac{\lambda_{i+1}^2 + \lambda_{i+2}^2 - \lambda_i^2}{2\lambda_{i+1}\lambda_{i+2}}. \tag{4}$$

This characterization allows us to construct P: it is the intersection point of the three corresponding arcs constructed on the sides of the triangle. These arcs indeed pass through one point because, assuming that (4) defines valid angles, it implies that $\phi_1 + \phi_2 + \phi_3 = 360°$.

What happens if the knot is dragged to a vertex, say A_1? In this case, $\vec{V}_1$ vanishes from the vectorial balance (2) since it has no component in the plane of the table. Then, the two remaining vectors $\vec{V}_2$ and $\vec{V}_3$ pull the knot away from the vertex A_1, and $\vec{V}_1$ "reappears", pulling the knot back to A_1. If the minimizing point is a vertex, it represents an unstable equilibrium of our mechanical system.

We are now ready to derive necessary and sufficient conditions for P to be an interior point of the given triangle.

1. **A necessary condition on the weights $\lambda_1, \lambda_2, \lambda_3$.**
   Relation (4) and the fact that $|\cos(\phi_i)| \leq 1$ lead to

$$(\lambda_{i+1} + \lambda_{i+2})^2 \geq \lambda_i^2, \quad (\lambda_{i+1} - \lambda_{i+2})^2 \leq \lambda_i^2. \tag{5}$$

   Rearranging (5) and ignoring the trivial cases of equality, we see that λ_i must satisfy

$$\lambda_{i+1} + \lambda_{i+2} > \lambda_i, \tag{6}$$

   that is, the λ_i must be the side lengths of a triangle. The angles of this triangle are $(180° - \phi_i)$.

   Constraint (6) can also be deduced by a geometric observation [**19**]. Suppose for example that $\lambda_1 \geq \lambda_2 + \lambda_3$. Then

$$\begin{aligned} \mathcal{F} &= \lambda_1 PA_1 + \lambda_2 PA_2 + \lambda_3 PA_3 \\ &\geq (\lambda_2 + \lambda_3)PA_1 + \lambda_2 PA_2 + \lambda_3 PA_3 \\ &= \lambda_2(PA_1 + PA_2) + \lambda_3(PA_1 + PA_3). \end{aligned}$$

   Using the triangle inequality we get $\mathcal{F} \geq \lambda_2 A_1A_2 + \lambda_3 A_1A_3$, which implies that the minimum is attained at the vertex A_1. This observation does not relate the constants λ_i to the angles ϕ_i.

2. A necessary condition on $\Delta A_1A_2A_3$.

Even if the weights satisfy the triangle inequality, it does not follow that P is an interior point. The additional condition on $\triangle A_1A_2A_3$ that must also be satisfied is $\phi_i > \alpha_i$. Expressing this condition in terms of the weights and the sides of $\triangle A_1A_2A_3$ we obtain the following system of inequalities that must be satisfied:

$$-\frac{\lambda_{i+1}^2 + \lambda_{i+2}^2 - \lambda_i^2}{2\lambda_{i+1}\lambda_{i+2}} < \frac{a_{i+1}^2 + a_{i+2}^2 - a_i^2}{2a_{i+1}a_{i+2}}. \tag{7}$$

Remark: The Fermat-Steiner problem is the special case where $\lambda_i = 1$. These equal weights satisfy the triangle inequality, so it is possible that P is an interior point. From (4) we have $\cos(\phi_i) = -1/2$, which tells us that P is the point from which the sides of the triangle are seen at 120°. Finally, (7) indicates that P is an interior point of $\triangle A_1A_2A_3$ if and only if the angles of the triangle are less than 120°; if one angle is greater than or equal to 120°, P is the vertex at this obtuse angle.

3. A TORRICELLI-STYLE SOLUTION. In this section we show how to extend the Torricelli-style construction to the weighted problem [**16**, pp. 113–118]. Let λ_i be the lengths of the sides of a triangle, and denote its angles by $\angle U$, $\angle V$, $\angle W$. Assume that we can construct a point P inside $\triangle A_1A_2A_3$ such that $\angle A_2PA_3 = 180° - \angle U$, $\angle A_3PA_1 = 180° - \angle V$, $\angle A_1PA_2 = 180° - \angle W$. Since $(180° - \angle U) + (180° - \angle V) + (180° - \angle W) = 360°$, it follows that P is the intersection point of the three corresponding arcs, constructed on the sides of $\triangle A_1A_2A_3$.

Through the points A_1, A_2, and A_3, respectively, draw the perpendiculars to PA_1, PA_2, and PA_3. The intersection points of these three lines determine a triangle $\triangle UVW$ whose angles are $\angle U$, $\angle V$, $\angle W$. Figure 4 (left panel) illustrates this auxiliary construction. We denote the lengths of the sides of $\triangle UVW$ by $VW = \mu \times \lambda_1$, $WU = \mu \times \lambda_2$, and $UV = \mu \times \lambda_3$ for some constant of proportionality $\mu > 0$.

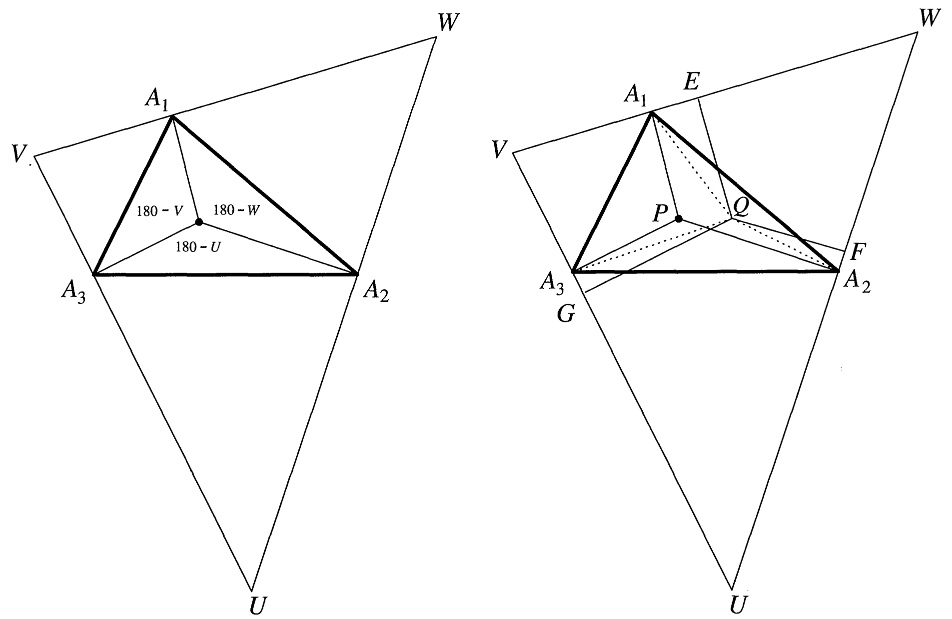

Figure 4. A Torricelli-style construction for the weighted Fermat-Steiner problem.

To prove that P minimizes $\mathcal{F}$ we must show that

$$\lambda_1 PA_1 + \lambda_2 PA_2 + \lambda_3 PA_3 < \lambda_1 QA_1 + \lambda_2 QA_2 + \lambda_3 QA_3$$

for any point $Q \neq P$. To this end, drop the perpendiculars QE, QF, QG on the sides VW, WU, VU of $\triangle UVW$, respectively (see Figure 4, right panel), and observe that

$$\begin{aligned}\frac{2}{\mu} S_{\triangle UVW} &= \lambda_1 PA_1 + \lambda_2 PA_2 + \lambda_3 PA_3 = \lambda_1 QE + \lambda_2 QF + \lambda_3 QG \\ &< \lambda_1 QA_1 + \lambda_2 QA_2 + \lambda_3 QA_3. \end{aligned} \tag{8}$$

The inequality in (8) follows from the observation that

$$QE \leq QA_1,\ QF \leq QA_2,\ QG \leq QA_3; \tag{9}$$

$Q \neq P$ implies that at least one of the inequalities in (9) must be sharp.

4. THE INVERSE FERMAT-STEINER PROBLEM. The solutions of the weighted Fermat-Steiner problem permit us to find the minimizing point for a given set of weights. This naturally sets the stage for the inverse problem: *Given a point $P \in \triangle A_1A_2A_3$, does there exist a unique set of positive weights λ_i, normalized by $\lambda_1 + \lambda_2 + \lambda_3 = 1$, for which P minimizes $\lambda_1 PA_1 + \lambda_2 PA_2 + \lambda_3 PA_3$? How can these weights be constructed?*

The answer to this inverse problem is positive, and here we combine the two solutions of the weighted problem to characterize and construct the desired weights.

Given $P \in \triangle A_1A_2A_3$, we see by writing $\angle A_{i+1}PA_{i+2} = \phi_i$ and using the mechanical solution that our inverse problem is equivalent to solving the system

$$-\frac{\lambda_{i+1}^2 + \lambda_{i+2}^2 - \lambda_i^2}{2\lambda_{i+1}\lambda_{i+2}} = \cos\phi_i \tag{10}$$

for the three unknowns $\lambda_1, \lambda_2, \lambda_3$. Now, the Torricelli-style solution provides a means for constructing the weights. Construct the triangle $\triangle UVW$ whose vertices are the intersection points of the lines through A_i that are perpendicular to PA_i. The angles of $\triangle UVW$ are, respectively, $180° - \phi_i$. Using (10) and the identity $\cos(180° - \phi_i) = -\cos\phi_i$, it follows by similarity that $\triangle UVW$ has sides proportional to λ_i. With an easy ruler and a compass construction we can divide the sides of $\triangle UVW$ by its perimeter and obtain the desired weights.

Finally, by applying the sine law to $\triangle UVW$, we conclude that $\lambda_1 : \lambda_2 : \lambda_3 = \sin(\phi_1) : \sin(\phi_2) : \sin(\phi_3)$. This provides an algebraic characterization of the desired weights, and completes the solution of the inverse problem.

5. CONCLUDING REMARKS. The Fermat-Steiner problem is a classical piece of mathematics. We have concentrated on the weighted variant of the Fermat-Steiner problem, and reviewed a mechanical and a geometrical solution. Both solutions are based on a surprising element. The mechanical approach earns its elegance by applying the minimum energy principle that helps us avoid messy calculus (compare with **[11]** and **[14]**). With this approach we can easily characterize the cases in which P lies inside the given triangle. The second method is a purely geometric solution that uses an auxiliary construction to solve the problem (compare with the geometric solution based on the weighted isogonic center in **[19]**). The combination of these approaches solves the inverse problem.

The Torricelli-style construction is a useful tool for further investigation. We conclude with some results that are obtained by constructing the auxiliary triangle for the orthocenter, incenter, circumcenter, and the center of gravity, and by using the inverse Fermat-Steiner problem. Proofs and details can be found in [**23**]. Propositions 1, 2, 3, and 4, for which we also compute $\mathcal{F}(P)$, establish three geometric inequalities.

Proposition 1. *If the angles of $\triangle A_1A_2A_3$ are acute, and $\lambda_1 = a_1, \lambda_2 = a_2, \lambda_3 = a_3$, then the point P is the orthocenter. In this case, $\mathcal{F}(P) = 4S$.*

Proposition 2. *If the weights are $\lambda_1 = I_2I_3, \lambda_2 = I_1I_3, \lambda_3 = I_1I_2$, where I_1, I_2, I_3 are the excenters of $\triangle A_1A_2A_3$, then the point P is the incenter and $\mathcal{F}(P) = 4Rp$.*

Proposition 3. *If the angles of $\triangle A_1A_2A_3$ are acute and $\lambda_1 = \sin 2\alpha_1$, $\lambda_2 = \sin 2\alpha_2$, $\lambda_3 = \sin 2\alpha_3$, then P is the circumcenter of $\triangle A_1A_2A_3$. In this case, $\mathcal{F}(P) = 4R \sin\alpha_1 \sin\alpha_2 \sin\alpha_3$.*

Proposition 4. *If $\lambda_1 : \lambda_2 : \lambda_3 = m_1 : m_2 : m_3$, where m_i is the median emanating from A_i, then the minimizing point P is the center of gravity of $\triangle A_1A_2A_3$. In the case where $\lambda_i = m_i$, $\mathcal{F}(P) = \frac{1}{2}(a_1^2 + a_2^2 + a_3^2)$.*

Proposition 5. *Suppose that the internal point $P \in \triangle A_1A_2A_3$ minimizes $\mathcal{F}(P)$ for the weights λ_i whose ratio satisfies $\lambda_1 : \lambda_2 : \lambda_3 = \sin\phi_1 : \sin\phi_2 : \sin\phi_3$. Then the weights λ_i^* with which the conjugate point P^* (i.e., the intersection point of the reflections, in the respective angle bisectors, of the three cevians that pass through P) minimizes $\mathcal{F}(P^*)$ are in the ratio $\lambda_1^* : \lambda_2^* : \lambda_3^* = \sin(\phi_1 - \alpha_1) : \sin(\phi_2 - \alpha_2) : \sin(\phi_3 - \alpha_3)$.*

REFERENCES

1. P. Cavallieri, *Exercitationes geometricae sex*, Bologna, 1647.
2. H. S. M. Coxeter, *Introduction to Geometry*, John Wiley, New York, 1961.
3. *Crux Mathematicorum* Problem 866, proposed by Jordi Dou in **9** (1983) 206; solution in vol. **10** (1984) 327–329.
4. R. Courant and H. Robbins, *What Is Mathematics?*, Oxford University Press, New York, 1951.
5. H. Dörrie, *100 Great Problems of Elementary Mathematics. Their History and Solution*, translated from German by D. Antin, Dover, New York, 1965.
6. N. Fuss, De minimis quibusdam geometricis, ope principii statici inventis, read to the Petersburg Academy of Sciences, Feb. 25, 1796.
7. I. Greenberg and R. A. Robetello, The three factory problem, *Math. Magazine* **38** (1965) 67–72.
8. E. Hofmann, Elementare Lösung einer minimumsaufgabe, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* **60** (1929) 22–23.
9. R. Honsberger. *Mathematical Gems, from Elementary Combinatorics, Number Theory and Geometry*, Mathematical Association of America, Washington, DC, 1993.
10. J. S. Mackay, Isogonic centers of triangles, *Proc. Edinburgh Math. Soc.* **XV** (1897) 100–118.
11. H. Mowaffac, An advanced calculus approach to finding the Fermat point, *Math. Magazine* **67** (1994) 29–34.
12. D. Pedoe, *Circles. A Mathematical View*, Mathematical Association of America, Washington, DC, 1995.
13. J. Pottage, *Geometrical Investigations. Illustrating the Art of Discovery in the Mathematical Field*, Addison-Wesley, London, 1983.
14. P. G. Spain, The Fermat point of a triangle. *Math. Magazine* **69** (1996) 131–133.
15. G. Steensholt, Note on an elementary poperty of triangles, *Amer. Math. Monthly* **63** (1956) 571–572.
16. H. Steinhaus, *Mathematical Snapshots*, Oxford University Press, New York, 1951.
17. T. Simpson, *Doctrine and application of fluxions*, 1750, items 36 and 431.
18. T. F. Tokieda, Mechanical ideas in geometry, *Amer. Math. Monthly* **105** (1998) 697–703.
19. J. Tong and Y. S Chua, The generalized Fermat's point, *Math. Magazine* **68** (1995) 214–215.
20. E. Torricelli, Opere di Evangelista Torricelli, edited by G. Loria and G. Vassura, vols. 1–3, Faenza, Montanari, 1919; vol. 4, Faenza, Lega, 1944.

21. V. Viviani, *Treatise De Maximis et Minimis*, 1659, Appendix, pp. 144–150.
22. W. J. van de Lindt, A geometric solution to the three factory problem, *Math. Magazine* **39** (1966) 162–165.
23. http://math.haifa.ac.il/~shay/steiner.htm

SHAY GUERON is a faculty member of the department of mathematics at the University of Haifa, Israel. His research interests are in applied mathematics (mathematical biology, and computations). He is also interested in problem solving and mathematical competitions, and has been the Israeli Team Leader for the International Mathematical Olympiad since 1994.
University of Haifa, Haifa, 31905, Israel.
shay@math.technion.ac.il

RAN TESSLER, a student of Shay Gueron, studies in a special gifted student enrichment program. He has won two silver medals in the International Mathematical Olympiad (2000, 2001) as a member of the Israeli team.
Herzelia, Israel.

An In Verse Function Theorem

Let g take an interval J,
Into an interval K.
Assume all the time,
Continu'us g'.
Pick a point inside J; call it a.

Look at $g'(a)$; we suppose,
It's invertible. From this it flows,
At least locally;
Bijective g,
And g^{-1} is smooth; so it goes.

Remark 1. We leave the problem of writing a proof in poetic form to the reader. This is, of course, an *In Verse Problem.*

Remark 2. We have purposely confined ourselves to a two-stanza format; our institution's administration looks most favorably upon publications that explicitly promote or exhibit *di-verse*-ity.

Remark 3. The content of Remark 2 notwithstanding, if we had succeeded in giving our In Verse Function Theorem in a single stanza, we would've sought publication in a more prestigous venue—say, one of the major *uni-verse*-ity publishing houses.

Submitted by
Louis A. Talman
Metropolitan State College of Denver