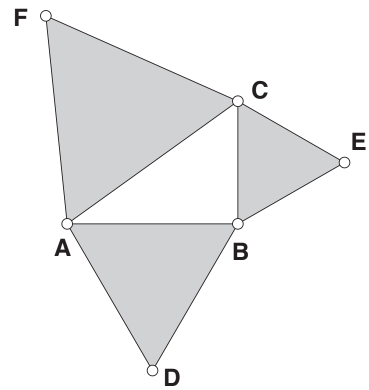
Name(s):

The Fermat-Torricelli Point

In this investigation, you'll make and prove some conjectures about a right triangle.

CONJECTURE

Open the sketch **Fermat 1.gsp.** Drag any vertex to investigate the shapes in the sketch.



1. What kind of triangles are constructed on the sides of right triangle *ABC*?

Construct lines *DC*, *EA*, and *FB*.

2. What do you notice about these lines? Drag any vertex of ΔABC to test your observations.

To measure the distance between two points, select both points and choose **Distance** from the Measure menu.

Measure the distances *DC*, *EA*, and *FB*.

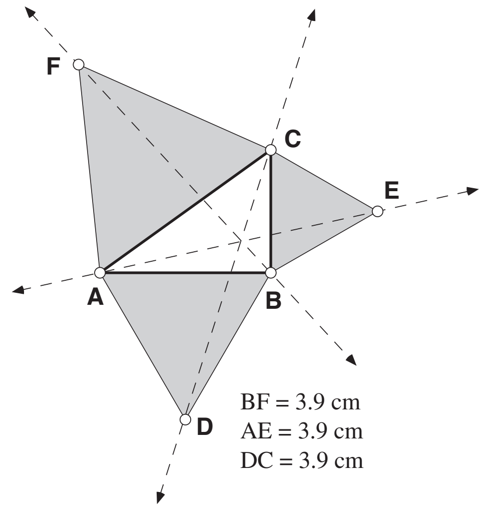
3. What do you notice about these distances? Carefully check your observations by further dragging.

4. Drag *C* past *B*. What happens to the triangles?

5. When you drag *C* past *B*, do your observations from Questions 2 and 3 still hold?

CHALLENGE Provide proofs of your conjectures from Questions 2, 3, and 5 above.

VERIFYING



Perhaps, though, there is another purpose to proof—as a testing ground for the stamina and ingenuity of the mathematician. We admire the conqueror of Everest, not because the top of Everest is a place we want to be, but just because it is so hard to get there.

—Davis and Hersh, 1983

You have noticed that if equilateral triangles *DBA*, *ECB*, and *FAC* are constructed on the sides of any right triangle *ABC*,

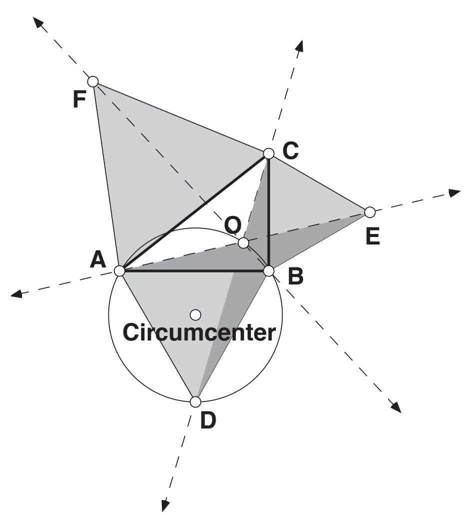
- The lines *DC*, *EA*, and *FB* are concurrent.
- Segments *DC*, *EA*, and *FB* are equal in length.

What is more, this result appears to be true even if the triangles lie inwardly. This point of concurrency is known as the Fermat-Torricelli point. (The mathematicians Pierre de Fermat and Evangelista Torricelli discovered it independently of each other.)

But how do we know our conjectures are really true? As you may have seen in the activity Concurrency, we must be careful not to form conclusions too easily. Let us investigate the problem further to come up with some ideas for a proof.

- Press the button that shows the interior of ΔDBC.
- Press the button that rotates the interior of ΔDBC around point *B* by $-60°$.

6. What do you notice about the rotated triangle? Try to find other pairs of triangles.



To construct a point at the intersection of three lines, select two of them and choose **Intersection** from the Construct menu.

To measure an angle, select three points on the angle, making sure the vertex is the middle selection. Then choose **Distance** from the Measure menu.

- Construct a point at the point of concurrency and label it *O*.
- Next, measure the six angles formed around point *O*.

7. What do you notice about the six angles around *O*? Drag a vertex of ΔABC to check your observations.

- Show the circumcircle of ΔADB.

8. What relationship is there between $\angle AOB$ and $\angle ADB$? What can you conclude from that? (*Hint:* Look at quadrilateral *AOBD*.)

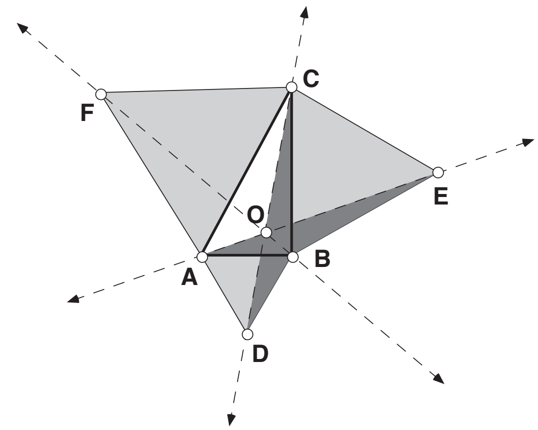
9. Press the button that shows all the circumcircles and circumcenters of the equilateral triangles. Look at the other two triangles. What do you notice?

CHALLENGE Try to use your observations from Questions 6–9 to construct a proof that $AE = BF = DC$, as well as that lines *AE*, *BF*, and *DC* are concurrent. Discuss your thoughts with a partner. If you get stuck, read the hints that follow.

PROVING SEGMENTS EQUAL

Here are some hints for planning a possible proof of this conjecture:

> If equilateral triangles *DBA*, *ECB*, and *FAC* are constructed on the sides of any right triangle *ABC*, then the lengths *DC*, *EA*, and *FB* are equal.



It will help if you hide the circumcircles and circumcenters. You don't need to hide triangles *DBC* and *ABE*, but remember that they are not part of the original construction.

10. In triangles *DBC* and *ABE*, what can you say about corresponding sides *DB* and *AB*? Why?

11. What can you say about corresponding sides *BC* and *BE*? Why?

12. What can you say about corresponding angles *DBC* and *ABE*? Why?

13. What can you therefore conclude about triangles *DBC* and *ABE*?

14. From Question 13, what can you conclude about corresponding sides *DC* and *AE*?

15. Repeat the above for triangles *EAC* and *BFC* to complete the proof.

16. Did your answers to any of the Questions 10–15 use the fact that $\angle ABC$ measures 90°? What does that imply about the conjecture you just proved?

17. Consider the quotation below in relation to your conclusion in Question 16.

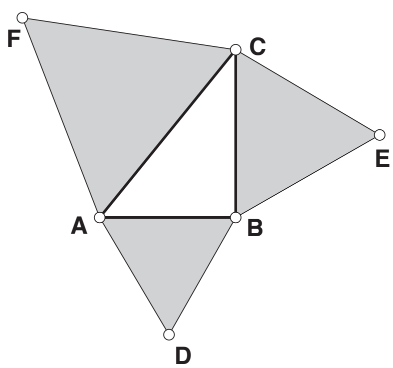
> *A good proof should convey an insight into exactly why the proposition is true. Such insight sometimes reveals the pleasant, unanticipated surprise that the proposition is merely a special case of a more general one, thus allowing for its immediate generalization.*
>
> —*M. D. de Villiers, 1998*

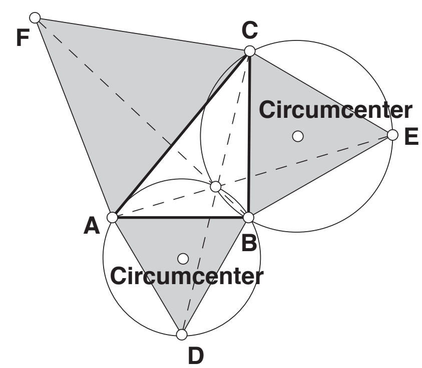
In what way has your proof provided you with insight that led to an immediate generalization?

PROVING LINES CONCURRENT

Here are some hints for planning a possible proof of your second conjecture:

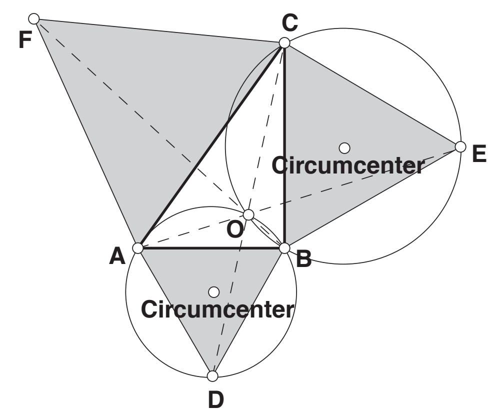
> If equilateral triangles *DBA*, *ECB*, and *FAC* are constructed on the sides of any right triangle *ABC*, the lines *DC*, *EA*, and *FB* are concurrent.





- Hide point *O* and the lines *BF*, *DC*, and *AE*.
- Press the button that hides triangles *DBC* and *ABE*.
- Press the button that shows circumcircles *ADB* and *BEC*. They should intersect at point *B*.
- Construct the other point of intersection of these two circles. This will be your new point *O*.
- Use the **Segment** tool (not the **Line** tool) to construct the six segments *OA*, *OB*, *OC*, *OD*, *OE*, and *OF*.

We will first prove that *AOE* and *DOC* are straight lines and then that the circumcircle *AFC* also passes through *O*. Using this fact, we will then show that *BOF* is also a straight line, which implies that $\overline{AE}$, $\overline{DC}$, and $\overline{BF}$ are concurrent at *O*. (*Note:* We *cannot* assume that lines *AOE*, *DOC*, and *BOF* are straight because that is what we need to prove.)



18. What can you say about the measure of angle *BCE*? Why?

19. From Question 18, what can you now say about the measure of angle *BOE*? Why?

20. What can you say about the measure of angle *BOA*? Why?

21. From Questions 19 and 20, what can you now conclude about angle *AOE*?

22. Repeat the same argument to show that *DOC* is a straight line.

23. From the angles determined above, calculate the measure of angle *AOC*.

24. From Question 23, what can you now conclude about quadrilateral *CFAO*? Why?

Press the button to show circumcircle *AFC* and check your result from Question 24.

25. Repeat the same argument as in Questions 18–21 to show that *BOF* is a straight line.

26. Would the preceding argument still be valid if $m\angle ABC$ were not 90°? What can you conclude from that?

27. Consider the quotation below, from a Russian mathematician, in relation to your conclusion in Question 26.

> *A good proof is one that makes us wiser.*
>
> *—Yu Manin, 1981*

In what way has the proof made you "wiser"?

Open the sketch **Fermat 2.gsp** and use it to check your conclusions in Questions 24 and 26.

Present Your Proofs

Create summaries of one or both of your proofs for any triangle. Your summaries may be in paper form or electronic form and may include a presentation sketch in Sketchpad. You may want to discuss these summaries with your partner or group.

Further Exploration

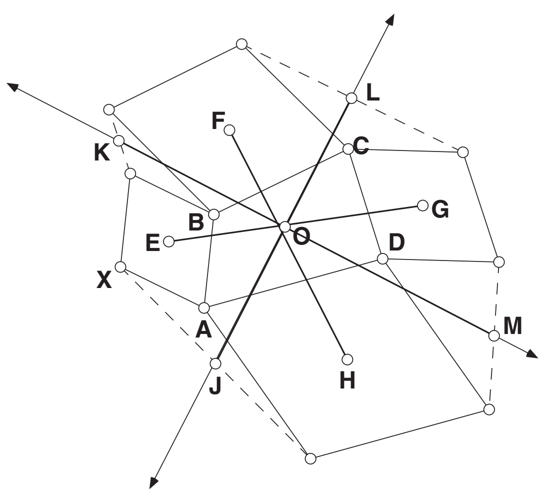
Can you find arrangements of similar or other triangles on the sides of any triangle *ABC* such that one or both of your results also hold?

In the second figure, $\overline{EG}$ is always congruent to $\overline{FH}$. Also, $\overline{KM}$ is perpendicular to $\overline{LN}$, where K, L, M, and N are the midpoints of the line segments joining adjacent vertices of the similar rhombuses as shown. A dynamic sketch is provided in **Aubel 2.gsp.** The "intersection" of these two results therefore yields van Aubel's theorem.

Two interesting special cases are obtained by constructing these similar rectangles and rhombuses on the sides of a parallelogram. In the first case, a rhombus is obtained, and in the second case, a rectangle. Proofs of these two special cases can be found in de Villiers (1996, 101–102).

All these results also nicely display the angle-side duality mentioned in the Teacher Notes for the Isosceles Trapezoid Midpoints activity, as well as in the Teacher Notes for the Logical Discovery: Circum Quad activity.

These two generalizations involving similar rectangles and rhombuses on the sides of any quadrilateral have since been generalized further to parallelograms, and to points other than the "centers" (see de Villiers 2000). A downloadable copy of this paper, as well as Sketchpad 3 sketches illustrating these generalizations, can be found on the author's Web site at http://mzone.mweb.co.za/residents/profmd.



m∠LOM = 90.000°
EG = 8.402 cm
FH = 8.402 cm

THE FERMAT-TORRICELLI POINT (PAGE 108)

This activity reinforces the function of proof discussed earlier, namely, logical discovery. After proving the results for a right triangle, students focus their attention on whether the arguments are still valid if angle *ABC* is not a right angle. This should make them realize that the result is immediately generalizable to *any* triangle. You can emphasize that this often happens in mathematical research, namely, that in proving some result, we find on reflection that some conditions were never used in the proof (i.e., were unnecessary) and that the result can therefore be generalized. The reason for starting with the right triangle is therefore to specifically illustrate this *discovery* function of proof.

Prerequisites: Knowledge of the properties of convex cyclic quadrilaterals (quadrilaterals that can be inscribed in a circle). Specifically, students should know that a convex quadrilateral is cyclic if and only if a pair of its opposite angles are supplementary. These properties have been discovered and proved in two earlier activities: Cyclic Quadrilateral and Cyclic Quadrilateral Converse. Also, students should be familiar with the SAS method of proving a pair of triangles congruent.

Sketch: Fermat 1.gsp. Additional sketches are **Fermat 2.gsp, Fermat 3.gsp,** and **Fermat 4.gsp.**

CONJECTURE

1. The "outer" triangles are all equilateral. If students are uncertain, encourage them to measure the sides or angles.
2. The lines *DC, EA,* and *FB* are concurrent.
3. The line segments *DC, EA,* and *FB* are equal in length.
4. The triangles lie inward.
5. Both results are still true.

CHALLENGE This gives students a first try at writing a proof for their conjectures.

VERIFYING

6. Triangle *DBC* maps onto triangle *ABE* (and they are therefore congruent).

7. The six angles around O all measure 60°.
8. They are supplementary, which implies that quadrilateral $AOBD$ is a cyclic quadrilateral (a quadrilateral that can be inscribed in a circle).
9. The circumcircles of the other two outer triangles also pass through O.

CHALLENGE After these hints, this challenge provides another opportunity for students to attempt to construct their own proofs.

PROVING SEGMENTS EQUAL

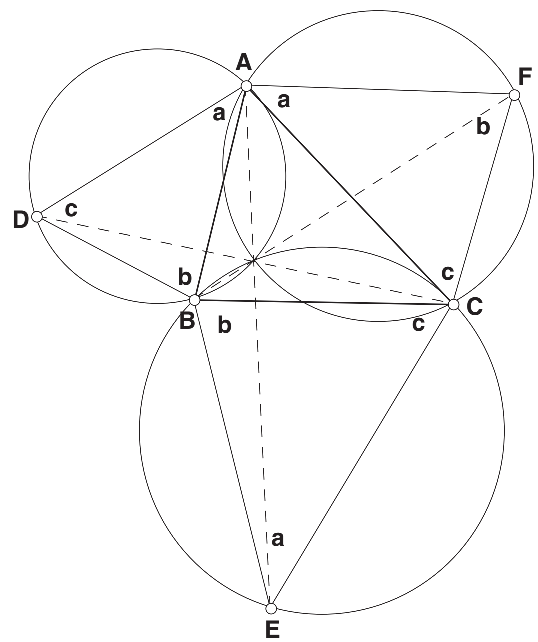
10. $DB = AB$, because triangle DBA is equilateral.
11. $BC = BE$, because triangle ECB is equilateral.
12. $m\angle DBC = 60° + m\angle ABC = m\angle ABE$.
13. Triangles DBC and ABE are congruent by SAS.
14. $DC = AE$, because corresponding parts of congruent triangles are congruent.
15. Similar to the above.
16. No. Therefore, the argument will still be valid even if $m\angle ABC$ is not 90°; the result is valid for *any* triangle.
17. Responses may vary. Students should notice that the argument they made to defend their conjecture did not require that angle ABC be a right angle. This means that their conjecture was in fact "a special case of a more general one."

PROVING LINES CONCURRENT

18. The measure of angle BCE is 60°, because triangle BCE is equilateral.
19. $m\angle BOE = m\angle BCE$ (angles on chord BE), and therefore $m\angle BOE$ is also 60°.
20. $m\angle BOA = 120°$, because it is supplementary to $\angle BDA$ of cyclic quadrilateral $DBOA$.
21. $m\angle AOE$ is 180°, because $m\angle BOA + m\angle BOE = 120° + 60° = 180°$.
22. Similar to the above.
23. $m\angle AOC = 360° - (m\angle BOA + m\angle BOC) = 360° - 240° = 120°$.
24. $CFAO$ is a cyclic quadrilateral (because opposite angles AOC and AFC are supplementary).
25. Similar to the above.
26. Yes, the argument would still be valid. Therefore, the result is valid for *any* triangle.
27. Responses may vary.

Further Exploration

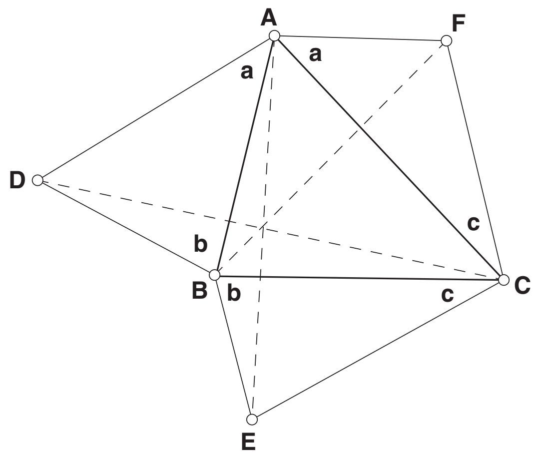
An interesting generalization of the concurrency result is the following (shown in the figure):



"If similar triangles DBA, CBE, and CFA are constructed outwardly on the sides of any triangle ABC, then segments DC, EA, and FB are concurrent."

The proof is similar to the one for the special case of equilateral triangles.

This result can be generalized even further as follows, and is shown in the next figure:



"If triangles *DBA*, *ECB*, and *FAC* are constructed outwardly on the sides of any triangle *ABC* so that $m\angle DAB = m\angle CAF$, $m\angle DBA = m\angle CBE$, and $m\angle ECB = m\angle ACF$, then segments *DC*, *EA*, and *FB* are concurrent."

One proof of this generalization, given in de Villiers (1996), is a little more complicated, using Ceva's theorem. A ready-made sketch called **Fermat 3.gsp,** which illustrates this generalization, is provided. An even further generalization is given and proven in de Villiers (1999a). This involves six triangles and a sketch is given in **Fermat 4.gsp.**

AIRPORT PROBLEM (PAGE 115)

This activity can be used to reinforce the idea of proof as a means of verification, because students are likely to experience some uncertainty as to the precise location of the optimal point. This activity can be done independently of the preceding activity, The Fermat-Torricelli Point (except for the Looking Back section, which specifically refers to the Fermat-Torricelli point). Furthermore, the solution of this problem does not require knowledge of the properties of cyclic quadrilaterals—a prerequisite for The Fermat-Torricelli Point activity.

If you are planning to do The Fermat-Torricelli Point activity, it is recommended that it precede the Airport Problem. Otherwise, the logical discovery of the generalization from a right triangle to any triangle in The Fermat-Torricelli Point activity may be less surprising.

Prerequisites: Students should have some familiarity with transformations, particularly rotations.

Sketch: Airport.gsp. Additional sketches are **Airport 2.gsp, Airport 3.gsp, Airport 4.gsp,** and **Airport 5.gsp.**

CONJECTURE

1. The angle measures are all approximately equal to 120°.
2. Same as Question 1.
3. The optimal point is situated where the measures of angles *ADC*, *BDA*, and *CDB* are all equal to 120°.
4. **Certainty:** Students may have some difficulty noticing that the three angle measures are equal and may not be entirely confident that this is really always the case. This doubt thus provides a good opportunity to again reinforce the idea of proof as a means of verification.

CHALLENGE Student responses will vary.

PROVING

5. $CD = CD'$, because they map to each other.
6. Triangle DCD' is equilateral ($m\angle D'CD = 60°$ and $CD = CD'$, which imply that $m\angle CD'D$ and $m\angle CDD'$ are also both 60°).
7. $D'D = DC$.
8. $AD = A'D'$, because they map to each other.