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Generalizing the Golden Ratio and Fibonacci[1]

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The Golden Ratio and Fibonacci

In a golden rectangle, the rectangle obtained by removing a square from one end is similar to the original rectangle (see Figure 1). The ratio of the length to the width of such a rectangle is called the *golden ratio* and is often denoted by the symbol ϕ. This ratio $\phi = a/b$ is therefore given by:

$$\frac{a}{b} = \frac{b}{a-b}.$$

Cross multiplying and then dividing by b^2 gives

$$\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right) - 1 = 0,$$

so the golden ratio is the positive root of the quadratic equation

$$x^2 - x - 1 = 0$$

and has the value $\frac{(1+\sqrt{5})}{2} \approx 1.61803$.

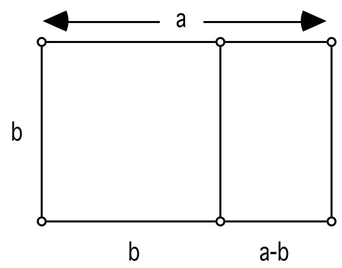

Figure 1

The well-known Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

[1] This article is a shortened version of an article “A Fibonacci generalization and its dual” published by the author in (2000). *Int. J. Math. Ed. Sci. Technol*, 31(3), Nov, 447-477.

can easily be constructed by the recurrence relation $F_n + F_{n+1} = F_{n+2}$, $F_1 = 1$, $F_2 = 1$, and where the n -th term is called F_n. Of course, any arbitrarily chosen F_1 and F_2 would do. A surprising result is the relationship of the Fibonacci sequence with the golden ratio:

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi .$$

Since convergence is fast, it is a good activity to compute these ratios using a calculator or a computer and watching them approach ϕ.

The Precious Metal Ratios

The Fibonacci sequence can be considered a special case of a whole family of sequences which can be constructed by simple variations to the above recurrence relation. For example, a general recurrence relation can be defined as $F_n + F_{n+k} = F_{n+k+1}$ where k is an integer so that $k \geq 0$. This implies that the first $k + 1$ terms can be arbitrarily chosen.

The following questions now seem natural to investigate further. Do the ratios of adjacent terms for this general family of sequences also approach limits? If so, are they also connected to the solution of some corresponding polynomial equations?

For example, consider the case $k = 2$ with the property $F_n + F_{n+2} = F_{n+3}$:

1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, ...

Here we have the following ratios (correct to four decimals):

$$\frac{F_{11}}{F_{10}} = \frac{28}{19} = 1.4736, \frac{F_{12}}{F_{11}} = 1.4642, \frac{F_{13}}{F_{12}} = 1.4634, \frac{F_{14}}{F_{13}} = 1.4666 \text{; etc.}$$

From the repetition of the first two decimals, we clearly already have convergence correct to two decimal places. This limit is the real solution of the equation $x^3 - x^2 - 1 = 0$.

In fact, each of these sequences is connected to a unique ratio. It seems appropriate to call this family of ratios the "precious metal" ratios with the golden ratio being one of them. It is a good exercise to construct some sequences of one's own in order to discover the generalization below, and convince oneself of its truth. Technology like graphics calculators with table facilities, or a spreadsheet on computer, could be very useful in this respect. In what follows, a partial proof of these observations is given.

Theorem

If F_n is the nth term of a sequence with the property: $F_n + F_{n+k} = F_{n+k+1}$, then for $k \geq 0$: $\lim_{n\to\infty} \frac{F_{n+k+1}}{F_{n+k}} = \alpha$ where α is the positive root of $x^{k+1} - x^k - 1 = 0$.

Partial Proof

If we assume that $\lim_{n\to\infty} \frac{F_{n+k+1}}{F_{n+k}} = \alpha$ exists, then we have the following:

$$F_{n+k+1} = F_{n+k} + F_n$$

$$\frac{F_{n+k+1}}{F_{n+k}} = 1 + \frac{F_n}{F_{n+k}}$$

$$\frac{F_{n+k+1}}{F_{n+k}} = 1 + \frac{F_n}{F_{n+1}} \bullet \frac{F_{n+1}}{F_{n+2}} \bullet \quad \ldots \quad \bullet \frac{F_{n+k-1}}{F_{n+k}}$$

$$\lim_{n\to\infty}\left(\frac{F_{n+k+1}}{F_{n+k}}\right) = 1 + \lim_{n\to\infty}\left(\frac{F_n}{F_{n+1}} \bullet \frac{F_{n+1}}{F_{n+2}} \bullet \quad \ldots \quad \bullet \frac{F_{n+k-1}}{F_{n+k}}\right)$$

$$\alpha = 1 + \frac{1}{\alpha^k}$$

$$\alpha^{k+1} - \alpha^k - 1 = 0$$

Thus, if $\lim_{n\to\infty} \frac{F_{n+k+1}}{F_{n+k}} = \alpha$ exists, then α is a root of the equation $x^{k+1} - x^k - 1 = 0$. Q.E.D.

To prove the existence of this limit in general is however a matter that goes beyond the scope of this article. For k is odd, the equation $x = 1 + \frac{1}{x^k}$ has two real solutions as easily seen graphically, and the approach used by [1] can be generalized. However, for k is even (where there is only one real solution as easily seen graphically), and the more general case which includes the consideration of complex roots, it appears that one would have to utilize an approach similar to that of [2].[2]

Students who explore it empirically may notice that these ratios α_k start at 2 for $k = 0$, and then appear to decrease towards a limiting value of 1 as k increases. This observation can also easily be explained as follows. For $k = 0$, the series has the rule $F_n + F_n = F_{n+1}$, obviously giving us the constant ratio $\frac{F_{n+1}}{F_n} = 2$, which of course corresponds to the solution of the equation $x = 1 + \frac{1}{x^k}$ for this value of k. By letting k increase in the latter equation, it now

[2] A complete proof can be found in an article by Sergio Falcon (2002) in *IJMEST*, and which can be downloaded directly from http://mysite.mweb.co.za/residents/profmd/fibonacci.pdf

Correct URL: http://dynamicmathematicslearning.com/fibonacci.pdf follows that $\frac{1}{x^k}$ decreases and therefore the root α must correspondingly decrease. Finally, taking the limit as $k \to \infty$ of the same equation, we obtain $\alpha = 1$.

It is also interesting to ask: what geometric interpretation can be given to these ratios α_k? Clearly if we start with a rectangle with sides a and b where $a \geq b$, then $\left(\frac{a}{b}\right)^{k+1} - \left(\frac{a}{b}\right)^{k} - 1 = 0$. Multiplying through by b^{k+1} and rearranging we obtain: $\left(\frac{a}{b}\right)^{k} = \frac{b}{a-b}$. Geometrically, this therefore means that after the square with sides b is removed, the rectangle obtained must be similar to a rectangle with sides a^k and b^k. Examples of corresponding rectangles for k = 0, k = 1 and k = 2 are respectively shown in Figures 2a, 2b and 2c. It is also obvious that as k increases, b approaches a and the rectangle tends towards a square.

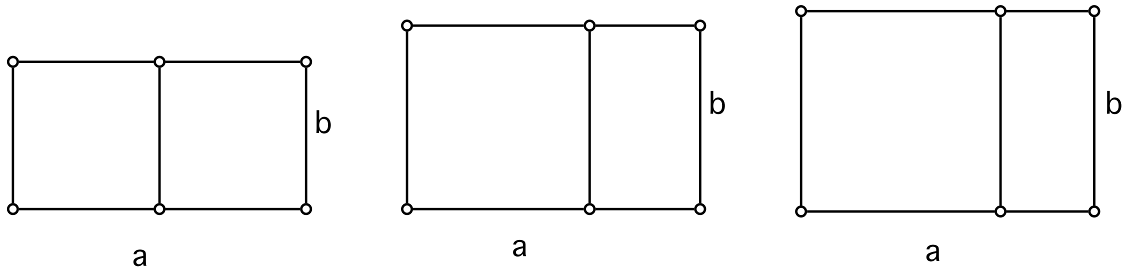

Figure 2

A Dual Family of Sequences

Lastly, it is also interesting to note that for each of the above sequences there exists a dual sequence with the recurrence relation $F_n \times F_{n+k} = F_{n+k+1}$ for $k \geq 0$, and corresponding ratio, $\lim_{n\to\infty} \frac{\log F_{n+k+1}}{\log F_{n+k}} = \alpha$, where α is the positive root of $x^{k+1} - x^k - 1 = 0$.

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2. Niven, I., Zuckerman, H. & Montgomery, H.L. *An introduction to the theory of numbers*, 5th Ed, Wiley, NY, (1991), pp. 493-499.