The Heuristic Beauty of Figural Patterns

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I am generally left with a sense of unease whenever I encounter questions of the type shown below:

Write down the next three terms in the following sequence: 2 ; 6 ; 12 ; …

My disquiet with such questions stems from the basic observation that any finite sequence of numbers can be justifiably continued in an infinite number of ways. Thus, any arbitrary choice of "the next three terms" should be given full credit. However, there seems to be a general acceptance that such questions are underscored by an unstated proviso that the three terms required should represent the "most obvious" or "most logical" sequence – whatever that's supposed to mean! Those questions that are prefaced, presumably with good intentions, with something along the lines of "Given that the following pattern continues in the same way…" are even more irksome, since there's an assumption that the "obvious" pattern is indeed that, i.e. "obvious", whereas I would argue that this notion is nonsensical. This can be understood by taking cognizance of the fact that a finite numeric sequence can be generated by an infinite number of functions, an idea that is readily supported by considering a finite number of points plotted in the Cartesian Plane where there would clearly be an infinite number of curves that could be drawn through the specified points. Thus, no finite sequence of numerical terms uniquely specifies the following term in a given sequence.

By way of example, let's return to our initial numeric sequence 2 ; 6 ; 12 ; … where only the first three terms have been specified. As argued above, there are an infinite number of ways to continue the sequence, and for each of these sequences there would be a corresponding general formula. One could argue that some sequences suggest themselves more readily than others, but even taking this argument into account still leaves the given sequence far from being uniquely specified. The pattern could, for example, continue based on a trivial repetitive cycle such as 2 ; 6 ; 12 ; 2 ; 6 ; 12 ; 2 ; 6 ; 12 ; etc. or 2 ; 6 ; 12 ; 6 ; 2 ; 6 ; 12 ; 6 ; 2 ; etc. If we assume that the sequence is based on a quadratic expression where there is a constant second difference, then we could generate the sequence 2 ; 6 ; 12 ; 20 ; 30 ; 42 ; etc. One could then use any number of standard techniques to arrive at the general term $T_n = n^2 + n$. One could arrive at an equivalent general term by noticing that $T_1 = 1 \times 2$, $T_2 = 2 \times 3$, $T_3 = 3 \times 4$ and thus arguing through inductive reasoning that the n^{th} term must be of the form $T_n = n \times (n+1)$. A different sequence could be arrived at by noticing that the 3$^{\text{rd}}$ term is the product of the two preceding terms, and thus argue that the general term could be expressed using the second-order recursive expression $T_n = T_{n-1} \times T_{n-2}$ where $T_1 = 2$, $T_2 = 6$ and $n \geq 3$. This would yield the sequence 2 ; 6 ; 12 ; 72 ; 864 ; 62 208 ; etc. Yet another possibility would be to assume the sequence is based on a cubic expression where the third difference is constant. Arbitrarily setting this constant third difference to equal 2, one can readily work backwards to give the sequence 2 ; 6 ; 12 ; 22 ; 38 ; 62 ; etc. This yields the general term $T_n = \dfrac{n^3 - 3n^2 + 14n - 6}{3}$.

There are of course many ways around this problem of ambiguity. The simplest solution is perhaps to present the numeric sequence in an unambiguous way, for example: “Given the following *quadratic sequence*, write down the next three terms: 2 ; 6 ; 12 ; …”. But of course this gives the game away, and the rest of the activity is likely to become nothing more than a rote exercise. An alternative way around the problem is to present the numeric sequence in a pictorial context, where the pictorial context itself suggests a deeper underlying structure. Number patterns presented in the form of a sequence of pictorial terms are more than simply a visual representation of a given numeric pattern. The critical difference between numeric and pictorial patterns is that, provided the pictorial context has been meaningfully understood, a pictorial representation is inherently less ambiguous than its isomorphic numeric counterpart.

Let us return to our original sequence of 2 ; 6 ; 12 ; … where only the first three terms have been specified, but this time let’s present these terms pictorially as shown in Figures 1-3.

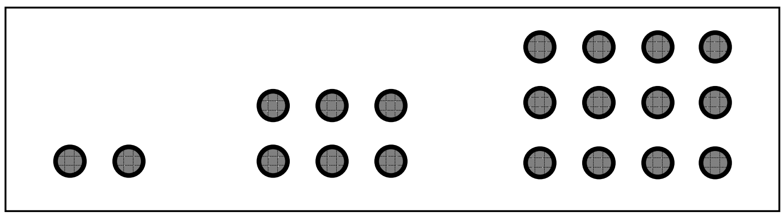

Figure 1 A growing sequence of rectangles made from dots

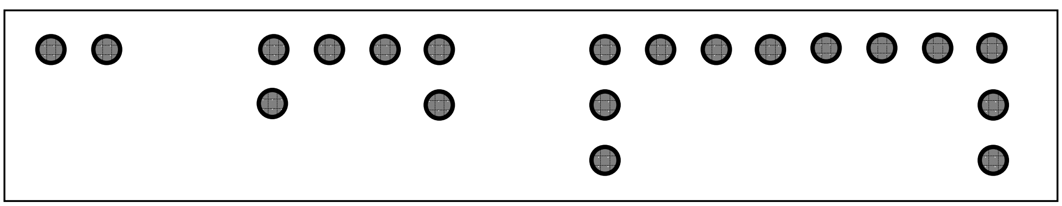

Figure 2 A growing sequence of “tables” made from dots

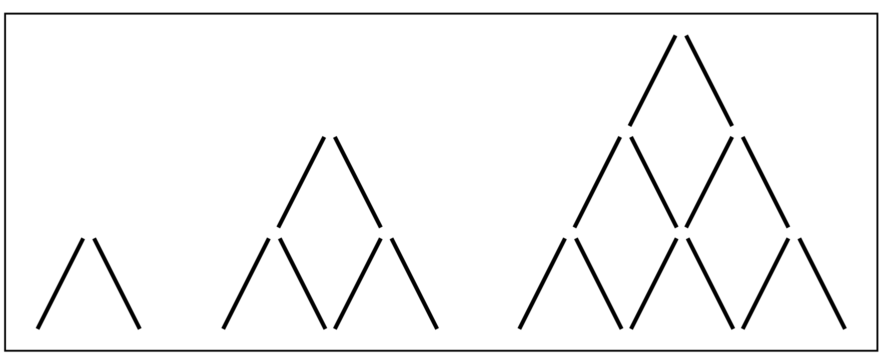

Figure 3 A growing sequence of mountain peaks made from matchsticks

For all three patterns the first three terms are numerically equivalent, *viz.* 2 ; 6 ; 12. However, the underlying structure suggested by the various pictorial representations yields sequences based on different expressions for the general term. Continuing the growing pattern of rectangles leads to the sequence 2 ; 6 ; 12 ; 20 ; 30 ; 42 ; etc. Continuing the growing pattern of tables leads to the sequence 2 ; 6 ; 12 ; 22 ; 40 ; 74 ; etc. while a continuation of the growing pattern of mountain peaks once again leads to the sequence 2 ; 6 ; 12 ; 20 ; 30 ; 42 ; etc.

Careful analysis of Figure 1 reveals that both the length and breadth of each successive rectangle are one more than the preceding rectangle. The dimensions of the three given rectangles are 1×2 , 2×3, and 3×4. The sequence thus continues with rectangles with dimensions 4×5, 5×6 and 6×7. The general expression for the sequence is thus $T_n = n\times(n+1)$. An analysis of Figure 2 suggests that the upper surface of the tables is growing exponentially (2 ; 4 ; 8 ; 16 ; 32 ; etc.) while the legs are increasing linearly, each leg growing according to the sequence 0 ; 1 ; 2 ; 3 ; etc. Combining these two elements yields the formula $T_n = 2^n + 2(n-1)$. Finally, an analysis of Figure 3 suggests that the structure comprises layers of matchsticks. The first term contains 1 pair of matchsticks, the second term contains $1+2$ pairs of matchsticks, and the third term contains $1+2+3$ pairs of matchsticks. Generalising this structure suggests that the n^{th} term contains $1+2+3+\ldots+n$ pairs of matchsticks and thus $2\left(\sum_{i=1}^{n} i\right)$ matchsticks in total. This yields the general term $T_n = n^2 + n$, which is algebraically equivalent to the formula arrived at for Figure 1. Importantly, however, both formulae were structured on different processes of visual reasoning.

The presentation of number patterns in a pictorial context has the potential to open up interesting spaces for classroom exploration and discussion. Of course, one can always reduce a given pictorial sequence to its isomorphic numerical equivalent, but the generalisation process then tends to become a somewhat superficial mechanical exercise, "an activity in its own right and not a means through which insights are gained into the original mathematical situation" (Hewitt, 1992, p. 7). The danger with such an approach is that the focus seems to be "the development of an algebraic relationship, rather than the development of a sense of generality" (Thornton, 2001, p. 252). As such, the general rule for the pattern becomes divorced from the scenario that gave rise to it. Such disconnected algebraic formulation neither illuminates the problem nor provides a means for validating the generated functional relationship (Noss, Healy & Hoyles, 1997). This becomes particularly problematic in situations where the justification of the general rule assumes significance (Byatt, 1994, p. 25). Indeed, as Hewitt (1992, p. 7) succinctly remarks, the problem with divorcing patterns of numbers from their original context is that any generalised statements become "statements about the results rather than the mathematical situation from which they came". Finally, as Orton (2004, p. 114) points out, justifying pattern generalisations is a legitimate and meaningful approach to proof, and provides pupils with valuable pre-proof experiences en route to more formalised mathematical proofs.

References

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