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Napoleon triangles and adventitious angles

MICHAEL FOX

In this article I investigate Napoleon triangles, generalisations of the mysterious equilateral triangle in Napoleon's theorem. I start with that theorem, develop some analogous results, find configurations with unexpected integer angles, and return to an extension of Napoleon's theorem. Many of the geometrical proofs depend upon spiral similarities, and the numerical work uses some unfamiliar trigonometrical identities.

A spiral similarity is a transformation combining a rotation with centre A and angle θ, say, and a dilatation with factor k having the same centre. This similarity, $A(\theta, k)$, transforms any figure into one that is directly similar. Any two directly similar figures are connected either by a dilatation (if their sides are parallel) or a spiral similarity. $A(\theta, k)$ followed by $B(\phi, l)$ is equivalent to $C(\theta + \phi, kl)$, where C is an invariant point in the joint transformation. Although we shall not need to find the centre of an arbitrary similarity, it is easily done. Given two points A, B and their images P, Q, let AB, PQ intersect at O. Then the points of intersection of circles APO, BQO are O and the centre of the similarity. Spiral similarities are discussed by Coxeter and Greitzer [1], and by Johnson [2], who calls them homologies.

I suppose Napoleon's theorem surprises everyone on the first encounter: the centres of equilateral triangles drawn outwards on the sides of an arbitrary triangle are themselves vertices of an equilateral triangle. To prove it, I use a slight adaptation. If triangle ABC has isosceles triangles PBC, QCA, RAB drawn outwards on its sides, each with base angles of 30°, then triangle PQR is equilateral. Although we can use trigonometry to show that the length of any side is symmetrical in a, b, c and so equals the others, I offer a geometric proof that can easily be generalised.

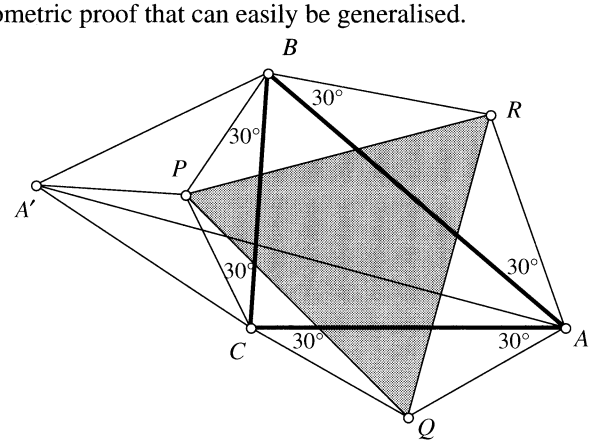


FIGURE 1

In Figure 1, triangle $A'CB$ is equilateral. The image of PQ under $C(30°, A'C/PC)$ is $A'A$, and the image of $A'A$ under $B(30°, PB/A'B)$ is PR. The scale factor $(A'C/PC).(PB/A'B) = 1$, and P returns to P after the double transformation, which is therefore equivalent to $P\ (60°, 1)$ — a pure rotation. Thus $PR = PQ$, $\angle QPR = 60°$, and triangle PQR is equilateral.

Now move P and A', keeping triangles PCB and $A'CB$ isosceles. Let their base angles be α and $\alpha + \beta$ as in Figure 2. Draw triangles CAQ and BAR similar respectively to $CA'P$ and $BA'P$. Then PQR will be isosceles with $PR = PQ$ and $\angle QPR = 2\beta$. The proof is simple: $C(\beta, AC'/PC)$ followed by $B(\beta, PB/A'B)$ takes PQ by way of $A'A$ to PR. So the combined similarity is $P(2\beta, 1)$, and the result follows. I call any triangle such as PQR, whose shape is independent of ABC, a Napoleon triangle.

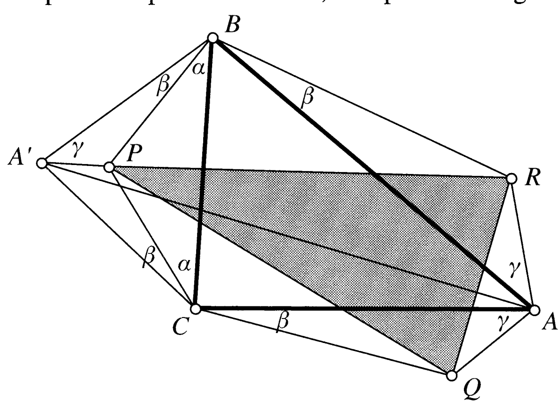


FIGURE 2

If we now remove the scaffolding, triangle $A'CB$, we have a result analogous to my version of Napoleon's theorem. Given an arbitrary triangle ABC, take points P, Q, R such that BAR, CAQ are similar triangles with base angles β, γ, and triangle BPC is isosceles with base angles α. Then, provided $\alpha + \beta + \gamma = 90°$, PQR is an isosceles triangle with angles 2β, $\alpha + \gamma$, $\alpha + \gamma$.

The case $\alpha = 15°$, $\beta = 45°$, $\gamma = 30°$ appears as a problem in Schumann and Green [3] who adapted it from the 1975 Mathematical Olympiad (see also [4]). Any configuration with $\beta = \alpha + \gamma = 45°$ will generate a right-angled isosceles triangle, and taking $\beta = 30°$, $\alpha + \gamma = 60°$ will give an equilateral Napoleon triangle. We may even set $\beta = 90°$, $\alpha = -\gamma$ to make P, Q, R collinear, with P the mid-point of QR. PQR is a degenerate Napoleon triangle.

We can go further, making the auxiliary triangle $A'CB$ and the position of P more general. Take triangle $A'CB$ and the point P with the angles shown in Figure 3. Draw CAQ and BAR similar to $CA'P$ and $BA'P$. Then we can find a spiral similarity that takes PQ to PR: for $C(s, A'C/PC)$ followed by $B(p, PB/A'B)$ takes PQ by way of $A'A$ to PR. So $\angle QPR$ is $p + s$. It is easier to repeat the process than to simplify the expression for the dilatation. Find C' so that BRC' is similar to BPC, then ABC' is similar to $A'BC$, and ARC' is similar to $A'PC$ and AQC. Apply $B(u, C'B/RB)$ then $A(r, RA/C'A)$ to RP, which becomes RQ. This shows that $\angle PRQ$ is $r + u$. Since $p+q+r+s+t+u = 180°$, $\angle RQP$ is $q + t$, and PQR is a Napoleon triangle, as its angles are independent of ABC. This proves that if, outside an arbitrary triangle ABC, we draw three triangles $A'BC$, $AB'C$, ABC' similar to each other (though not necessarily to ABC), and take corresponding points P, Q, R in the three triangles, then PQR is a Napoleon triangle with $\angle QPR = \angle QCA + \angle ABR$, etc.

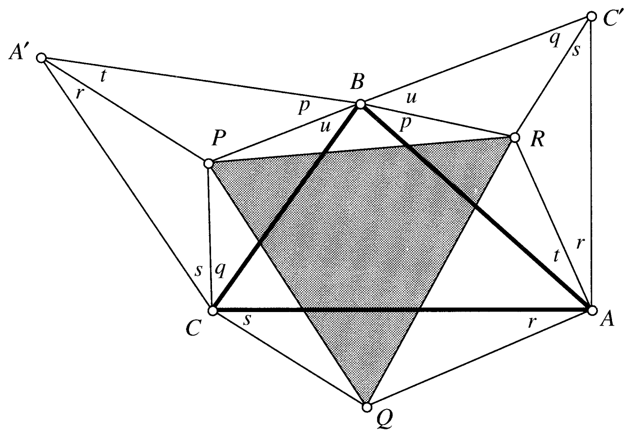


FIGURE 3

Since triangle ABC is arbitrary, we can deform it. For instance we can move B across the line AC; triangles PBC, QCA, RAB (and $A'BC$, etc.) will overlap ABC, but PQR will still be the same shape (although with the opposite sense). Our results therefore hold in these cases as well.

Can we find integer values in degrees for p, q, r, s, t, u, v? They satisfy

$$p + q + r + s + t + u = 180°. \tag{1}$$

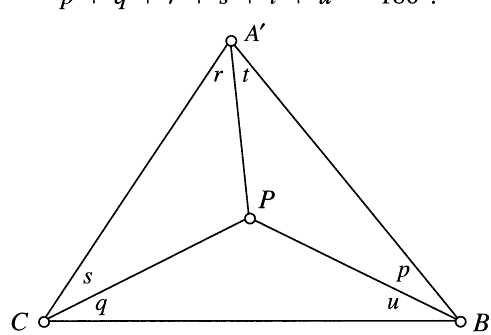


FIGURE 4

If we use the sine rule in the three small triangles in Figure 4, we find

$$\frac{\sin u}{\sin q} \cdot \frac{\sin s}{\sin r} \cdot \frac{\sin t}{\sin p} = \frac{PC}{PB} \cdot \frac{PA'}{PC} \cdot \frac{PB}{PA'} = 1,$$

$$\text{thus} \qquad \sin p \sin q \sin r = \sin s \sin t \sin u. \tag{2}$$

If we are given any four of these angles we can find the other two. For if we know p, q, r, s, then (1) gives the value of $t + u$, and (2) becomes $\sin t \sin u = \sin p \sin q \sin r / \sin s$. We can express $\sin t \sin u$ as $[\cos(t - u) - \cos(t + u)]/2$, and so find the value of $t - u$, from which we obtain t and u. If instead we know p, q, s, t, then we set

$$\frac{\sin r}{\sin u} = \frac{\sin s \sin t}{\sin p \sin q}.$$

We know the value of $r + u$: let us call it v. So $r = v - u$, and

$$\frac{\sin s \sin t}{\sin p \sin q} = \sin v \cot u - \cos v.$$

We find $\cot u$, which gives u and r.

Once we have a set of angles $p, q, r; s, t, u$ satisfying (1) and (2), we can obtain several configurations of A', B, C, P. We must use each angle once; the marked angles in each small triangle must come one from each triple, and the two parts of angles A', B, C must also be one from each triple. So the solution $\alpha, \beta, \gamma; \alpha, \beta, \gamma$ used in Figure 2 can be rearranged so that $A'BC$ is no longer isosceles, and P may be the incentre, circumcentre or orthocentre of $A'BC$. The angles of PQR will, of course, change; but PQR will still be a Napoleon triangle with integral angles. Because of these possible rearrangements we can use a standard form for listing solutions of (1) and (2). I shall take $p \leqslant q \leqslant r$, $p \leqslant s \leqslant t \leqslant u$; and if $p = s$, then $q \leqslant t$.

To find non-trivial integral solutions of (1) and (2) I had to resort to a computer search. This could only give possibilities: sets of values within, say, 10^{-5} of an integer. Each trial solution had to be verified for its exactness. I restricted the search by taking P inside $A'CB$, and looking for solutions in standard form. Even so, the computer tested over 600,000 sets of values to come up with 123 possible distinct solutions, all of which proved to be exact. Of these, 85 are in four families; the other 38 being adventitious. The families are

A.	$\theta,\ 30 - \theta,\ 90 + \theta;$	$2\theta, 30 - \theta, 30$	$1 \leqslant \theta \leqslant 14$
B.	$\theta,\ 30 - \theta,\ 60 + \theta;$	$\theta, 30, 60 - 2\theta$	$1 \leqslant \theta \leqslant 29$
C.	$\theta,\ 30,\ 90 - 3\theta;$	$2\theta, 30 - \theta, 30 + \theta$	$1 \leqslant \theta \leqslant 29$
D.	$\theta,\ 60 - 4\theta,\ 60 + \theta;$	$3\theta, 30 - 2\theta, 30 + \theta$	$1 \leqslant \theta \leqslant 14$

These give 85 distinct solutions rather than 86, since 10, 20, 70; 10, 30, 40 is in B and D, with $\theta = 10$ in each.

The 38 adventitious solutions are listed in the table.

2,	34,	94;	10,	16,	24.	6,	18,	78;	6,	24,	48.
2,	46,	62;	8,	12,	50.	6,	18,	84;	12,	12,	48.
3,	18,	93;	6,	12,	48.	6,	21,	57;	9,	12,	75.
3,	24,	81;	6,	15,	51.	6,	30,	66;	12,	18,	48.
3,	30,	75;	9,	12,	51.	6,	30,	78;	18,	24,	24.
3,	30,	93;	12,	18,	24.	6,	42,	48;	12,	18,	54.
3,	39,	66;	9,	15,	48.	6,	42,	54;	12,	24,	42.
3,	39,	75;	9,	24,	30.	9,	15,	75;	9,	18,	54.
3,	39,	81;	15,	18,	24.	9,	15,	81;	12,	15,	48.
3,	42,	75;	12,	21,	27.	9,	15,	87;	12,	18,	39.
3,	48,	54;	9,	15,	51.	9,	18,	69;	12,	15,	57.
3,	48,	69;	15,	18,	27.	9,	33,	54;	15,	21,	48.
3,	54,	63;	15,	21,	24.	10,	26,	70;	22,	24,	28.
4,	38,	64;	10,	18,	46.	12,	12,	84;	12,	18,	42.
6,	6,	126;	6,	12,	24.	12,	24,	66;	18,	30,	30.
6,	9,	99;	6,	12,	48.	12,	30,	48;	18,	18,	54.
6,	12,	96;	6,	18,	42.	12,	33,	39;	15,	18,	63.
6,	12,	114;	12,	18,	18.	15,	24,	57;	18,	27,	39.
6,	15,	105;	12,	18,	24.	18,	24,	54;	24,	30,	30.

Some of them are tricky to prove. However, there are three identities which are useful.

1. We can expand the products of sines into sums. The six angles total 180°, so we have $\sin(p+q+r) = \sin(s+t+u)$. Consequently, $\sin p \sin q \sin r = \sin s \sin t \sin u$ is equivalent to $g(p,q,r) - g(s,t,u) = 0$, where $g(p,q,r) = \sin(-p+q+r) + \sin(p-q+r) + \sin(p+q-r)$.
2. $\sin(60° + \theta) - \sin(60° - \theta) \equiv \sin\theta$.

   This is easily proved.
3. $\sin 54° - \sin 18° \equiv \sin 30°$.

In Figure 5, triangles BDE, BDA and ADE are isosceles, giving $BE = BD = DA$. Also EF is perpendicular to AD, so $DF = \frac{1}{2}DA$. Thus

$$BE(\sin 54° - \sin 18°) = CF - CD$$
$$= DF = \tfrac{1}{2}BE,$$

whence $\sin 54° - \sin 18° = \frac{1}{2} = \sin 30°$. (We also have $2BE \sin 18° = DE$, and $DE \sin 54° = DF$, proving that $\sin 18° \sin 54° = \frac{1}{4}$. Therefore $-\sin 18°$ and $\sin 54°$ are the roots of the equation $4x^2 - 2x - 1 = 0$.)

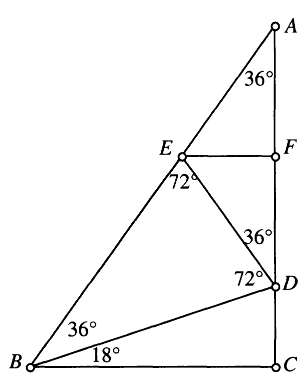


FIGURE 5

As an example let us verify that

$$\sin 10^\circ \sin 26^\circ \sin 70^\circ = \sin 22^\circ \sin 24^\circ \sin 28^\circ.$$

Using identity 1 we try proving instead that

$$(\sin 86^\circ + \sin 54^\circ - \sin 34^\circ) - (\sin 30^\circ + \sin 26^\circ + \sin 18^\circ)$$

vanishes. The identity 3 reduces this to $\sin 86^\circ - \sin 34^\circ - \sin 26^\circ$, which we can write as $\sin(60^\circ + 26^\circ) - \sin(60^\circ - 26^\circ) - \sin 26^\circ$. The second identity shows that this is zero. So 10, 26, 70; 22, 24, 28 is a valid solution.

We have seen that any solution, whether or not in standard form, makes PQR a Napoleon triangle with angles $p + s$, $q + t$, $r + u$. We find a surprising result if we rearrange the solution 6, 42, 48; 12, 18, 54, as 6, 42, 48; 54, 18, 12. This gives a new configuration where PQR is equilateral (Figure 6); a result new to me.

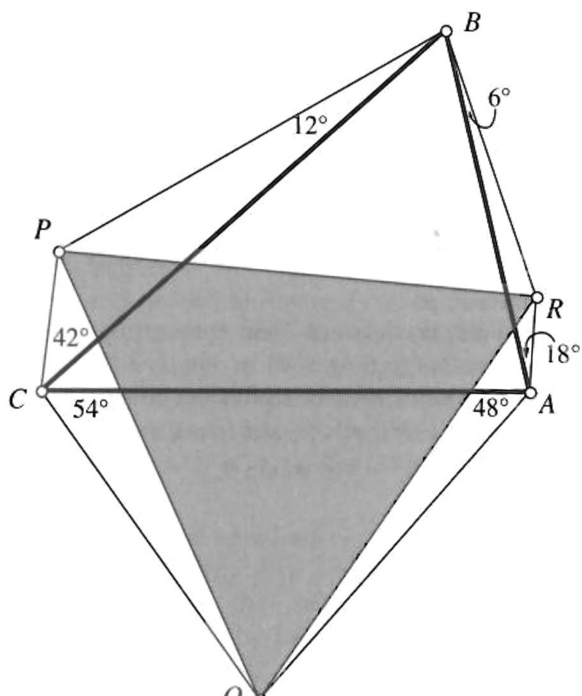


FIGURE 6

So far I have taken P inside triangle $A'CB$; but why should it not be outside? Fortunately we can deal with this by transforming the solutions that we already have. There are two cases, depending on whether P is in a region marked I or II (Figure 7).

If P is in I we reinterpret an existing solution (Figure 8). The sign convention is that angles measured towards or past an adjacent side (such as $u + p$ in the right-hand diagram) are positive; those measured away from an adjacent side (such as p) are negative. We must take care at the new A', so

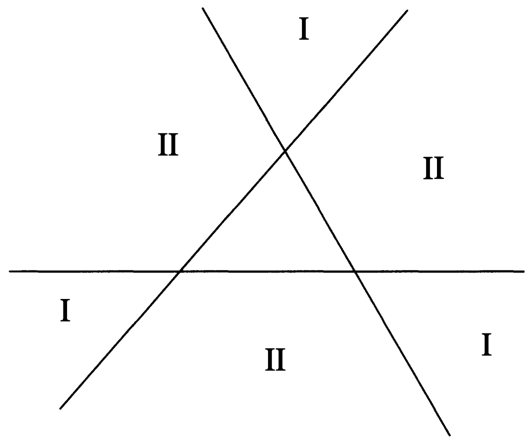


FIGURE 7

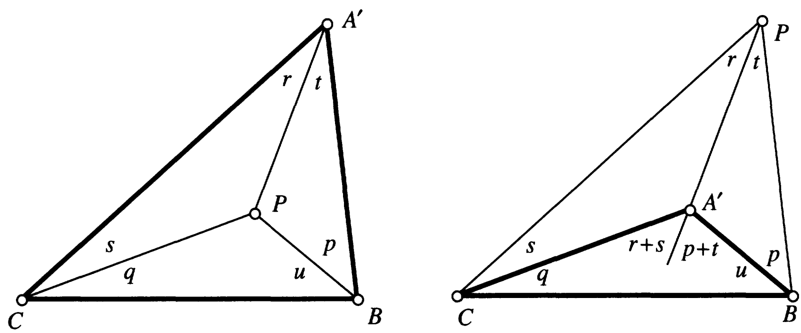
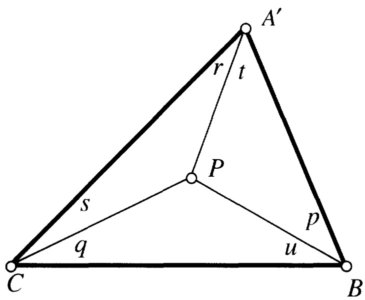


FIGURE 8

that the six angles total 180°. This gives $-p, q + s, r + s; -s, t + p, u + p$ as a new solution. If we identify the new vertices differently, or start with a different arrangement of p, q, r; s, t, u, we obtain other solutions. Thus 10, 20, 70; 10, 30, 40 transforms into –10, 30, 80; –10, 40, 50. If we rearrange the starting values as 30, 10, 40; 10, 70, 20 we get instead –30, 50, 20; –10, 50, 100, which is –30, 20, 50; –10, 50, 100 in standard form.

If P lies in II we can transform our basic triangle $A'CB$. Apply $A'(t, A'B/A'P)$ to triangle $A'PC$, obtaining $A'BP'$; and $A'(-r, A'C'/A'P)$, to $A'PB$, giving $A'CP'$ (Figure 9) — this gives the same point P'. With our sign convention we find that $p, q, r; s, t, u$ gives $180 - (p + t), -(q + t), r; 180 - (s + r), -(u + r), t$. We can relabel the transformed diagram to find other solutions, or rearrange the angles before the transformation. For example, 10, 20, 70; 10, 30, 40 gives 140, –50, 70; 100, –110, 30; which in standard form is –110, 30, 100; –50, 70, 140. The rearrangement 30, 10, 40; 10, 70, 20 gives a new solution whose standard form is –80, 40, 80; –60, 70, 130. The diagram for this solution is a remarkable quadrilateral (Figure 10) containing the first nine multiples of 10°.



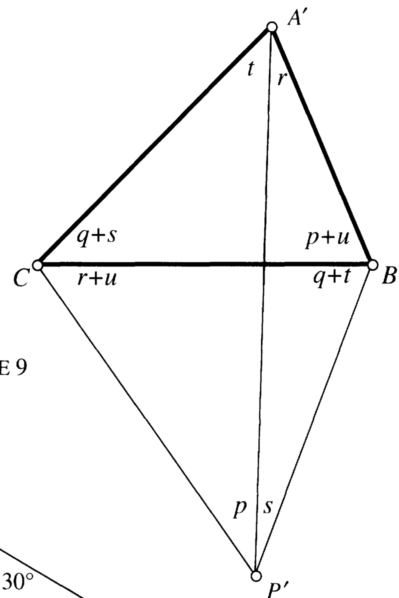


FIGURE 9

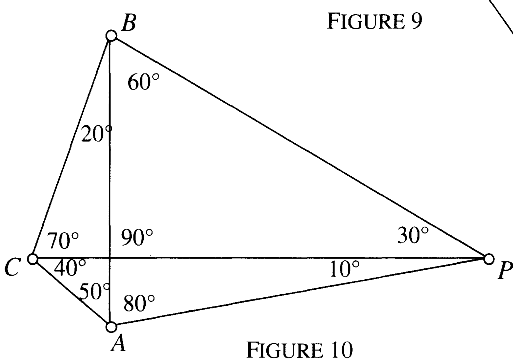


FIGURE 10

With these transformations, every basic solution with P inside ABC gives rise to many distinct solutions with P outside. Each of these, applied to an arbitrary triangle ABC, gives a configuration where at least one of PBC, QCA, RAB overlaps ABC, but PQR is still a Napoleon triangle. We now have every possible integer configuration, for if there were others, we could apply the inverse of one of the transformations and arrive at a new basic solution with P inside ABC. But there are no other such solutions.

We have seen that any integer-angled isosceles triangle can be a Napoleon triangle. How can we determine what others there are? If we are given an arbitrary Napoleon triangle PQR, can we find triangles PBC, QCA, RAB with integer angles?

To answer this we need another result. Apply $P(180° - (r + s), PC/PA')$ to $PA'B$, transforming it into PCA''; and $P(180° - (p + t), PB/PA')$ to $PA'C$, which becomes PBA'' (Figure 11). The angles A'', C, B are $p + s$, $q + t$, $r + u$. Now apply corresponding transformations to $B'CA$ and $C'AB$, obtaining $B''CA$, $C''AB$, but leaving PBC, QCA and RAB unchanged (Figure 12). Triangle PQR still has angles $p + s$, $q + t$, $r + u$, which are also the angles of $A''CB$, $CB''A$, BAC''. We therefore have the generalisation of Napoleon's theorem, given by Wells [5]: on the sides of an arbitrary triangle ABC draw triangles $A''CB$, $CB''A$, BAC'' similar to each other (though not necessarily to ABC). Let P, Q, R be corresponding points in each, then PQR is similar to each of the three triangles. (The three similar triangles drawn round ABC are orientated differently from those in Figure 3: the centres of the similarities linking them are not the vertices A, B, C.)

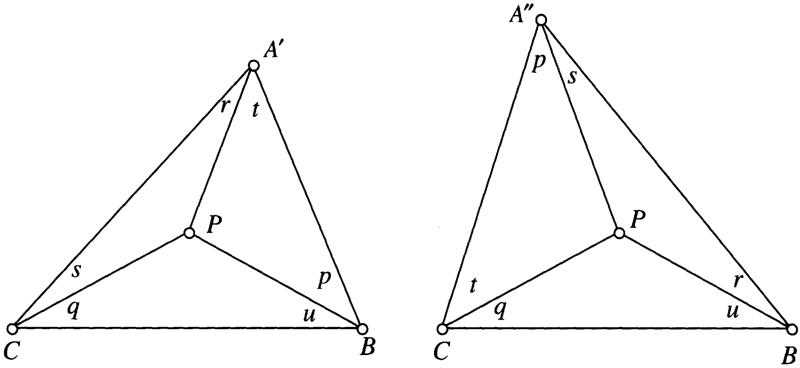


FIGURE 11

It follows that, if a Napoleon triangle has integral angles α, β, γ, then so has $A''CB$. If we can find P so that $A''P$, BP, CP make integral angles with the sides of $A''CB$, then the triangles PBC, QCA, RAB will have integral angles. In fact we can take any triangle as a Napoleon triangle, provided we allow one degenerate case.

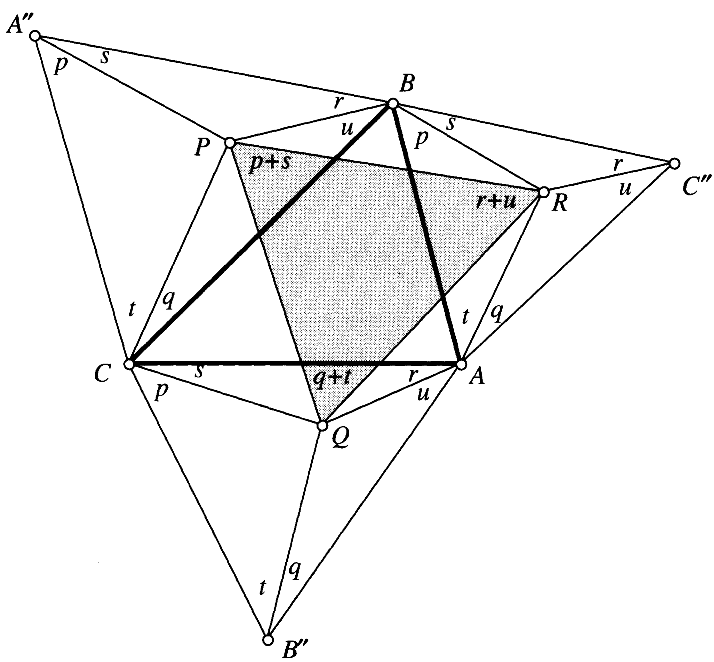


FIGURE 12

First, suppose PQR is not right-angled. We can take P as the circumcentre or orthocentre of $A''CB$. If P is the circumcentre we have a simple configuration where PBC, QCA, RAB are isosceles with base angles $90° - \alpha$, $90° - \beta$, $90° - \gamma$ respectively.

If all the angles of PQR are even, we may let PQR be right-angled, and take P as the incentre of $A''CB$. If PQR is right-angled and scalene with two odd angles, we have to let one of the external triangles, say PCB, degenerate into a straight line. Possibly the easiest solution is to take P as the circumcentre of $A''CB$. If the right angle is at P then A'' is also a right angle, and P is at the midpoint of BC.

All these are based on the apparently trivial solutions of the form $p, q, r; p, q, r$. For certain triangles we can find other configurations based on the special families or the adventitious solutions. However, there seems to be no simple way, given an arbitrary PQR, of determining whether such a configuration exists.

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