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How can dynamic geometry environment assist learning of geometrical proofs at the university level?

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Abstract: This paper describes an experiment in teaching Euclidean geometry which was undertaken by the author at a Canadian University. The approach combined the methodology of basic geometric configurations (BGC) with the introduction of a dynamic geometry environment (DGE). BGC is a geometrical drawing that depicts a statement along with auxiliary elements pertinent to its proof. Benefits of using BGCs in teaching geometry were enhanced by employment of corresponding applets produced with dynamic geometry software. Several problems that highlight geometrical invariance observable in DGE are presented. Responses of 13 students who had taken the course indicate a potential of this practice and suggest directions for further research.

Introduction

Many North American universities have an undergraduate course in *Euclidean Geometry*. This subject is essential for students pursuing engineering and science degrees as well as for those who preparing to teach in a grade school. The course usually places great emphasis on proofs and reasoning, and hence presents a challenge especially for students with weak geometrical backgrounds and insufficient retention of high school knowledge in plane geometry. Students resist making sense of the many geometrical facts and have a tendency to memorize most of the material without internalizing it. As a result, many students struggle instead of enjoying a subject that is essentially vibrant and alive.

Instructors in Euclidean Geometry may consider incorporating dynamic geometry environments (DGE) hoping that students would appreciate the beauty of the subject given an opportunity to experiment and explore. At the same time it is unlikely that a DGE is going to enhance the learning of geometry just by its presence. A thoughtful design of the classroom activities and homework practices is required.

This paper resulted from a small educational project aiming to address this need while teaching a course in Euclidean Geometry in a Canadian university (Kondratieva, 2011c). The paper starts with a brief discussion of existing literature regarding challenges and various approaches to teach proofs in geometry. The role of DGE in connection with educational research on the development of deductive thinking is highlighted in the next section. With this in mind, the following section talks about the possibility to blend an approach based on the use of basic geometric configurations with construction of interactive applets in DGE. Then I give examples of several geometrical problems introduction of which was reshaped by the presence of DGE. The problems are used to illustrate various advantages that the focus on invariance observation in a DGE may have for the learner. These problems also represent well the approach which was undertaken in teaching the course. Students' responses to this new teaching initiative are given in the last section.

1. The challenges and approaches to teach geometrical proofs.

This paper concerns teaching geometry at the undergraduate university level. The first course aims to introduce students to the axiomatic world of Euclidean Geometry and improve their ability to reason logically and compose rigorous proofs. Equipped with these abilities students sharpen their precision of thinking in terms of definitions, theorems and abstract mathematical concepts, which is vital for learning in all branches of mathematics (Rav 1999; Balacheff 2010). At the same time, while learning to prove students also enrich their repertoire of techniques and approaches applicable in problem solving situations (Hanna & Barbeau 2010).

However, learning to prove and think rigorously presents a major challenge for the students and consequently requires attention from mathematics educators at all levels. "Pupils fail to appreciate the critical distinction between empirical and deductive arguments and in general show a preference for the use of empirical argument over deductive reasoning." As well "proof is not 'used' as a part of problem-solving and is widely regarded by students as an irrelevant, 'added-on' activity" (Hoyles & Jones 1998, p. 121; see also Coe & Ruthven (1994)).

Naturally, the process of cognitive development of a learner requires a long time before they start to conceptualize their empirical experiences and symbolic exercises in terms of formal objects and operations and before their thought become hypothetical and fully abstract (Tall et al). By encountering various properties of an object and establishing (and proving!) relations of implications between these properties, one comes to a mental construction of "crystalline concepts" corresponding to the objects of study. This scenario "offers the possibility of increasingly complex and connected knowledge structures" (Tall 2011, p 6.) The process of cognitive growth in this direction requires, beside time and effort on behalf of the learner, specific "mathematical activities that could facilitate the learning of mathematical proof" (Balacheff 2010, p. 133). When learning proofs in geometry, one would benefit from "problem situations calling for an interaction between visual methods and geometrical methods" (Laborde 1998, p.114). A productive way of incorporating experimentation and proving needs to be found so that "*proofs do not replace measurements but make them more intelligent*" (Janhke 2007, p.83). At the same time, it was observed that between the Platonic world of Euclidean geometry and learners' world of physical practice and experiences (such as drawing or paper folding) there exists an "experimental-theoretical gap" (Lopez-Real and Leung, 2006, p.667) which originates from a fundamentally different nature of those two worlds. These differences may also be expressed in terms of dichotomies characterizing the focus in teaching geometry: intuition-deduction, construction-proof, and spatial-numerical (Laborde 1985).

Thus we are confronted with the following dilemma: the development of deductive thinking is our goal (in this undergraduate course) but the elements of formal thinking should grow from learner's intuition and prior experience and can not be simply imposed on a learner in its final form (Freudenthal, 1971). Learners who are exclusively exposed to formal procedural approach in teaching geometry (such as two column proof) often experience "epistemological anxiety" (Wilensky, 1993) resulted from not being able to understand the meaning and purpose of the actions they perform, even if they receive high marks for their performance. "Meaning" is also regarded by Sfard (2003) as one of the main ten learners' needs.

This dilemma may be resolved by making careful distinction between visual appearances and structural organizations of geometrical images (see also Kondratieva, 2011d). Leading students to pay attention on significant elements of a geometrical construction and interrelation between these elements we help them to transform "messy drawings" into "figural concept" that were defined by Fishbain (1993) as "mental entities which possess simultaneously conceptual and figural characteristics" (p.143). While geometry largely relies on pictorial materials, figures support visual thinking only if a learner grasps the mathematical structure they represent (Arnheim 1969). Understanding in geometry "cannot be achieved just through visual evidence as understanding requires restructuring the system of conceptions and ideas. Proof based on theoretical arguments becomes a means to understand." (Laborde 2000, p. 155). Indeed, proofs can help a learner to *explain* certain empirical observations at the informal deduction level. But other important functions of proof such as *systematization* and *construction of a theory* (De Villiers 1990; Hanna 2000) must not be forgotten by mathematics instructors.

Jahnke (2007) proposes that "inventing hypotheses and testing their consequences is more productive … than forming elaborate chain of deductions" (p.79). However, these practices also require from students to possess certain mathematical culture and must be directed towards a unifying mathematical framework. Students "should build a small network of theorems based on empirical evidence" and become accustomed to "*hypothetico-deductive method* which is fundamental for scientific thinking" (p.83). Similarly, Tanguay & Grenier (2010) suggest that, "current curricular trends, promulgating proving processes based on experimentation and conjectures, will lead to an effective learning of proof, with proof attaining its full meaning in the learners' understanding only if these processes are set within a genuine process of building 'small theories' ". The conjectures should be formed and viewed a part of "hypothetico-deductive networks, which would then be confronted with the initial experimentations" (p.41). As well, "*incorporation* of well-known facts into a new framework" will call for a proof functioning as means of "*construction* of an empirical theory" (Hanna 2000, p. 8).

2. Proofs in dynamic geometry environments.

Intuition required in the process of making conjectures and inventing hypotheses develops through students' experiences not only in formal logical manipulations but also in experimental explorations of objects and ideas (De Villiers 1990). Constructions with a compass and straightedge were traditionally used for building students' geometrical intuition. Since the time when first dynamic geometry environments such as Cabri or Geometer Sketchpad became available for students, educators started to look at various possibilities these systems can offer to the learner of geometry. At first the systems were used to produce accurate and nicely looking geometrical drawings. Soon after, it was realized that *dragging* operation available in these environments contains a much larger potential than just creating a number of static cases of a certain geometrical property. An observed "motion dependency" can be interpreted by a learner and transformed into a logical conditional relation within a mathematical context (Mariotti 2006). Dragging allows one "to "see" mathematical properties so easily" that some educators feared that this fact "might reduce or even kill any need for proof and thus any learning of how to develop the proof" (Laborde 2000, p.151). Consequently it can widen the gap between the inductive nature of experimental geometry (enhanced by this dynamic feature) and deductive nature of Euclidean geometry (Mason 1991). Considering students inclination to exclusively experimental verification of geometrical statements, this scenario of narrow-minded use of DGE in schools still presents a real threat to a proof-oriented curriculum. However, it was found that certain pedagogical approaches which incorporate DGE in the right way may in fast facilitate learning of proofs and help students to produce logical links between various properties of a dynamic drawing. First, DGEs give a new dynamic meaning to static statements of Euclidean geometry. For example, the phrase “any point on a circle” can be interpreted as a physical action of dragging a point along the circle. Thus theorems can be illustrated by constructing dynamic drawings satisfying the set of conditions listed in the theorem and then observing the facts listed in the concluding part of the theorem. For example, one can *observe* that “three side perpendicular bisectors of any acute triangle always intersect at one point which lies inside the triangle” by drawing an acute triangle with perpendicular bisectors to its sides and them dragging the vertices of this triangle so that it remains acute. Such an experimentation-observation process may help the student to notice the implicative structure (as opposed to “and”-structure when properties are viewed simultaneously with no relational grasp between them) of the statement and perhaps to memorize the statement better, but it may never illuminate the reason of the observed phenomenon.

Prefabricated drawings allowing students to drag points in a constrained domain and observe the result of such dragging, are known as *robust constructions* (Healy, 2000). They had been found very useful specifically for the purpose of illustrating geometrical statements and letting students to make a clear distinction between premises and conclusions of these theorems. But it is another type of dynamic drawings that currently gives hope to mathematics educators in connections with learning to prove. *Soft constructions* (Healy, 2000) allow a learner to conjecture the region for dragging of an element of a drawing that leads to production of the desired property. For example we ask the student “under what condition do the three perpendicular bisectors intersect at a point lying inside the triangle?” The student then experiments by dragging vertices of a triangle and makes a conjecture based on the experimental evidence that “the triangle must be acute”. This type of dragging is called “maintaining” since it purposely maintains the property of interest (Baccaglini-Frank & Mariotti, 2010). In contrast with the robust construction described above, in a soft model the student conjectures the conditions themselves as they can also explore the case when the triangle is not acute. This wider context for exploration may also lead to additional by-product conjectures such as “the intersection point of perpendicular bisectors lies precisely on the side of the triangle when the triangle is right-angled”. These and other features of soft constructions identified through classroom research hold a promise that “maintaining dragging” may produce argumentation leading to both the conjecture and proof and thus help to bridge the worlds of experimental and theoretical geometries (Baccaglini-Frank & Mariotti, 2010).

3. Dynamic geometry and basic geometric configurations

There are various ways of employment of DGE in mathematics classrooms. In more conservative types, dynamic geometry is used as “a convenient parallel to paper and pencil: to provide accurate static figures and generate measurement data; to highlight invariant properties through their visual salience under dragging" (Ruthven et al 2007, p. 299). Some teachers may use the same problems and theorems as in the conventional curriculum but allow students "to consider geometrical relationship inductively before being exposed to deductive proof" (Lampert 1993, 150). In his approach largely supported by a DGE, textbook or worksheet can provide an important structuring resource for lesson activity. In more progressive constructivist scenario teachers accept that "learning might take place in computer-based situations without reference to a paper-and-pencil environment" (Laborde, 2001, p.311) and without regarding a book or teacher as a main source of information. In the latter case students assume a greater ownership for their study when they learn from experimenting with dynamical figures. At the same time, left alone in a DGE the students may undergo various pitfalls such as (1) invent their own terminology or assign different meaning to standard terminology (2) stick with certain techniques that proved useful and keep returning to them despite availability of a better alternatives; (3) focus on procedures rather that on analysis of geometrical structure; (4) may not appreciate the significance of invariance (Jones, 1999). Thus students need teacher's intervention in their practice with a DGE in order to guide their learning.

In our approach the content was strictly shaped by the book but at the same time students were encouraged to learn using several resources combining the book material, lecture discussions as well as experimentation in the dynamic environment. All three sources of geometrical ideas were welcomed by the instructor. In fact the study was interested to what extend the students will use this freedom in their learning.

My traditional approach to teaching Euclidean Geometry emphasizes the use of *basic geometric configurations* (BGCs) - fundamental geometric facts expressed in drawing (Kondratieva 2011a). Such drawings contain auxiliary elements and labels (equal angles, equal segments, perpendicular and parallel lines) that allow remembering the statements along with the ideas of their proofs. For example, Figure AA (middle) shows a BGC corresponding to the fact that "locus of vertex of a right triangle with hypotenuse FG is the circle with diameter FG". For comparison, Figure AA (left) illustrates possible visualization of this fact in a DGE by setting a segment AB of a fixed length, lines AD and BC to be perpendicular, and then dragging point D with *Trace* of point C turned on.

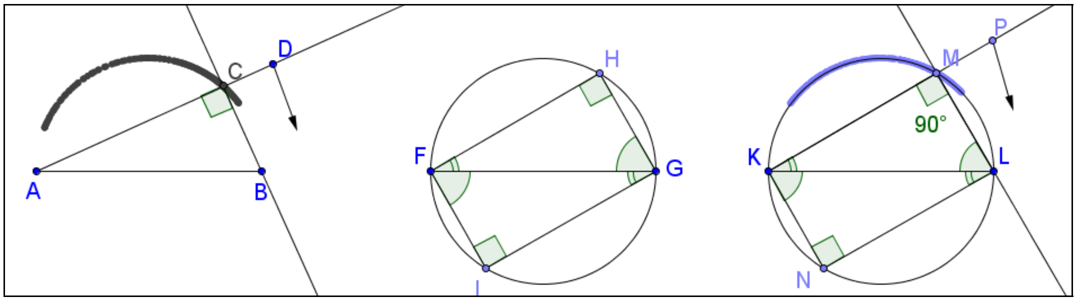

Figure AA. Property illustrated by dragging (left), corresponding BGC (middle) and its version in DGE(right).

Idealistically, BGC are images of *crystalline concepts*, since they embrace properties of a geometrical object in a structural way. BGCs are the stepping stones to proving or solving geometric problems. Once a BGC is identified by a learner as a part of more complex structure, the learner can activate one of the implications decoded by this BGC and thus make a deductive step in her reasoning. Thus, learning BGCs in geometry is similar to learning an alphabet of pictorial language. An employment of BGC approach calls for the following teacher's actions : (1) Asking students to explain what relations they observe in a figure and what they think about the role of the auxiliary lines drawn on the original figure. (2) Constantly showing connections to already learned geometrical facts and focusing students' attention on the key ideas used in a particular solution. (3) Demonstrating several proofs or solutions of the same problem in order to show connections between geometry, trigonometry and algebra. (4) Directing students' attention to the implications, converse and equivalence of statements. (5) Helping students summarize their findings in the form of a mathematical statement. (6) Surprising students with an unexpected conclusion or asking them to correct errors in a flawed reasoning (Kondratieva 2009, 2011a).

Taking into account that "computers can offer a new context for designing innovative activities to address the main problem of linkage between empirical experiments and deductive reasoning" (Osta 1998, p 111), I introduce my students to dynamic drawings of BGCs (applets) produced in GeoGebra (GG). These applets mostly present robust constrictions and allow observing the elements of the statement or proof that are invariant under dragging (see for example, Figure AA (right). The applets are used during the lectures discussions to accompany the blackboard presentation by "dynamical visual proofs, which are based on 'drawing in movement' that can be properly performed in a dynamical environment" (Gravina 2008). The applets are linked to the webpage associated with the course and are available for students' further experimentations. This way, students become accustomed to the idea of supporting their geometrical reasoning by an interaction with the dynamic drawings.

During the course, students were asked to perform the following assignments. They were asked to create a robust dynamic drawing based on a verbal description and a static figure in the book. They were asked to recognize BGCs as a part of a proof given in the book and to illustrate this proof by constructing their own applets. Finally, students were asked to create their own proofs of given problems and indicate BGCs employed. In the latter case they could choose to create a related applet first and try to explain behavior of the drawing and produce a proof based on these explanations. However, they could also choose to work with pencil-and-paper only.

The novelty of this approach consists in combining the methodology of the BGC approach with the advantages offered by the geometry software, in order to balance empirical and deductive practices. First, students read and analyze sample proofs and identify BGCs and key ideas pertinent to the proofs. At the same time students construct interactive applets in GG with the requirement to make the constraints described in the statement indestructible by dragging. This forces them to use geometrical properties of the object they draw. Students are asked to show auxiliary lines and measurements pertinent to the idea of the proofs. Students are encouraged to invent alternative proofs to the statements they analyze and interpret with the help of GG. Students are given examples of all these activities in class. They discuss BGCs with their teacher using both static and dynamic drawings. As the semester evolves, the students are provided with fewer hints for problems and are asked to continue building GG applets and experiment with them in order to find their own solutions. In this way students gradually adopt the Euclidean (synthetic) geometry tradition of proofs and learn to recognize and apply BGCs. The students learn to observe and explain individual empirical facts, then build, and check their "small theories" based on many dynamic and static drawings.

In a DGE "a critical point of the solving processes is the visual recognition of a geometrical invariant by the students, which allows them to move to geometry." (Laborde 1998, p. 120). Next section explains several possible advantages for students' learning that are related to the phenomenon of invariance with respect to dragging.

4. Case-invariant solutions in a DGE.

When doing "proof by cases" on paper very often we need to come up with different ideas and techniques in each particular case of a more general situation. However, in a DGE a smooth visual transition between different cases is often available. For example, one may easily pass from the case of obtuse triangle to the case of acute triangle by dragging a vertex of this triangle. In this section we are interested in solutions which are valid in all possible cases of a given problem and their case-invariance is observable by dragging. We discuss four examples of problems from Euclidean geometry and their case-invariant solutions produced in a DGE.

However, in each of our examples the discussion of a case-invariant solution has a slightly different emphasis. In the first example we demonstrate the importance of consideration of special cases which could be much simpler to handle than the general case. The key contraction that was found in a special case of our problem suggested the solution to the original problem taken in full generality. In the second example we highlight the advantage of Trace function available in DGE and how its use may generate the insight in the solution which is valid in various situations. The third example illustrates the possibility to learn some additional geometrical facts useful for proving other statements while looking at different cases of a theorem's proof. The last example shows that working with various cases of a problem in a DGE allows one to deeply understand certain geometrical notions (such as area) and make connections with other branches of mathematical knowledge.

4.1. Case-invariant solution originated from a simple special case.

Very often it is relatively straightforward for students to make an applet with required conditions, and the property of interest becomes easily observable by dragging base points of a drawing. At the same time the dragging does not produce a significant insight in students towards a possible proof of the observed fact. It is of interest to us what can be done in such a case in order to help the learners to generate some useful ideas. One thing to do is to drag the points to produce special cases of the general situation. Special cases may suggest a way helpful for a generalization (Polya, 1945). They also may present an interesting problem by itself. Consider the following example.

Problem 1 (Shawyer 2010, p. 133). *Suppose that ABCD is a convex cyclic quadrilateral. Let points P, U, Q, V be the midpoints of arcs AB, BC, CD, and DA respectively. Prove that PQ is orthogonal to UV (see figure XX, left).*

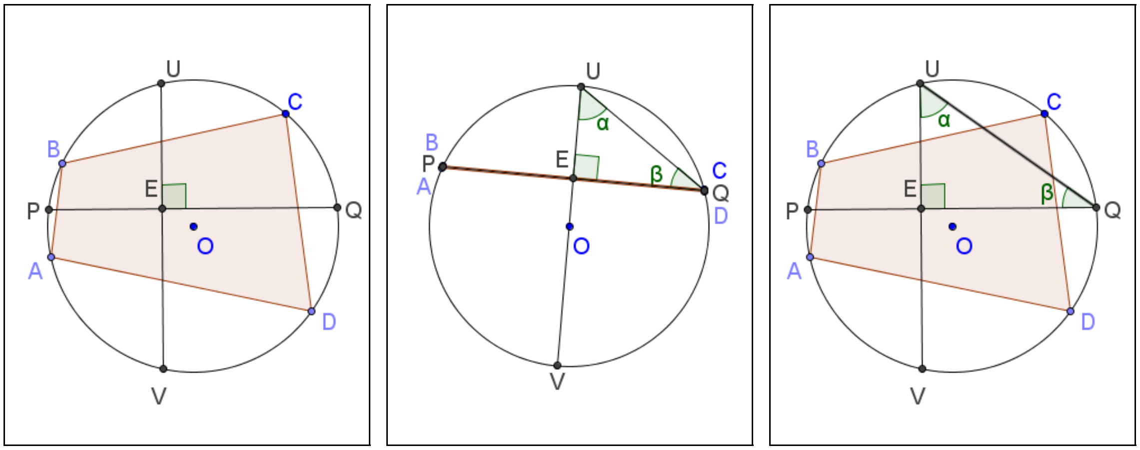

Figure XX. Problem 1: general case (left), special case (middle), and the idea of the proof.

Once the applet is built, the statement can be illustrated by dragging vertices A,B,C,D and observing the angle UEQ. In particular, we can drag the points to produce a special case, when points A, B, and P coincide, as well as points C, D and Q (see figure XX, middle). The good thing is that we are now dealing with only 4 point, P,U,Q, and V such that arcs PU and UQ are equal as well as arcs QV and VP are equal. Denoting their length by x and y respectively, we obtain 2x+2y=360, and thus x+y=180. Notice that we have inscribed angle PQU subtended by arc PU measured x degrees. Similarly, inscribed angle QUV subtended by arc QV measured y degrees. Recalling that an inscribed angle is exactly half of the corresponding central angle, or equivalently, of the corresponding arc measure, we conclude that PQU + QUV=(x+y)/2=90. Thus triangle UEQ is a right triangle, and the statement is proved. In considering this special case, we introduced in the construction an auxiliary segment UQ and focused on inscribed angles α and β. Our key algebraic observation used the fact that certain pairs of arcs are equal and altogether they constitute a full circle. Now by dragging points A and D back to produce a general case we observe that these ideas remain useful. Once again, we focus on triangle UEQ and angles α and β. We denote equal arcs as follows: AP=PB=a, BU=UC=b, CQ=QD=c, DV=VA=d. In this case $\alpha = (QD + DV)/2 = (c+d)/2$ and $\beta = (PB + BU)/2 = (a+b)/2$. On the other hand we can see that $2a + 2b = 2c + 2d = 360^0$, and thus $\alpha + \beta = 360^0 / 4 = 90^0$, which completes the proof in the general case.

Note that the solution to this problem outlined in (Shawyer, 2010) is based on the geometry of complex numbers and this presents an opportunity to compare the two approaches and to strengthen the various mathematical connections. If the special case of the problem is discussed in class, students will have a good chance to discover the general case solution and complete this assignment at home on their own.

4.2. Case-invariant solution suggested by the locus observation in DGE.

Our second example is a problem where students must first find the locus of points and then explain their answer.

Problem 2 (Shawyer 2010, p. 133). *Let C be a circle and P be any fixed point. Consider the collection of all lines on P that intersect C. Suppose that typical such line meets the circle at points A and B. Find the locus of mid-point of AB.*

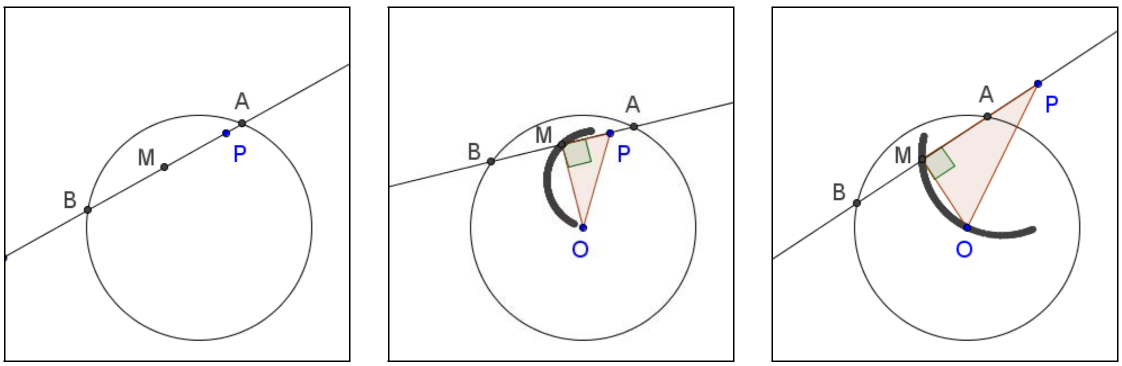

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Figure YY. Problem 2: static drawing, possible cases with Trace On and the idea of the proof.

A static figure (Figure YY, left) can be easily drawn for this problem, but students do not find it very helpful. By dragging with the *Trace* function of the midpoint M turned on we observe that the locus forms a circle with diameter PO, where O is the center of circle C (see Figure YY, middle). As we drag P inside or outside of the circle C, we observe that M follows the arc of the circle with diameter PO (Figure YY, right). The explanation of this fact comes from recognition of two basic geometric configurations learned in the course. First, MO is orthogonal to AB because it is a radius-chord property. Second, since PMO is a right triangle, M lies on the circle with the diameter equal to the hypotenuse of this triangle. Once again, this idea works in all cases regardless of whether P lies inside or outside the circle C.

It was found during our teaching experiment that seeing the locus as a result of the dragging action triggered in the majority of students the recognition of the "inscribed right triangle" configuration and consequently generated insight into the proof (see section 5.5 for more details).

4.3. Case-invariant proof and notice of additional geometrical facts.

Our next example refers to the Six Point Circle theorem. The theorem states that *in any triangle the midpoints of the sides and feet of the altitudes lie on a circle*. One possible proof is based on the following approach. One needs to recognise that the quadrilateral formed by the three midpoints and one foot is an isosceles trapezoid and thus is cyclic.

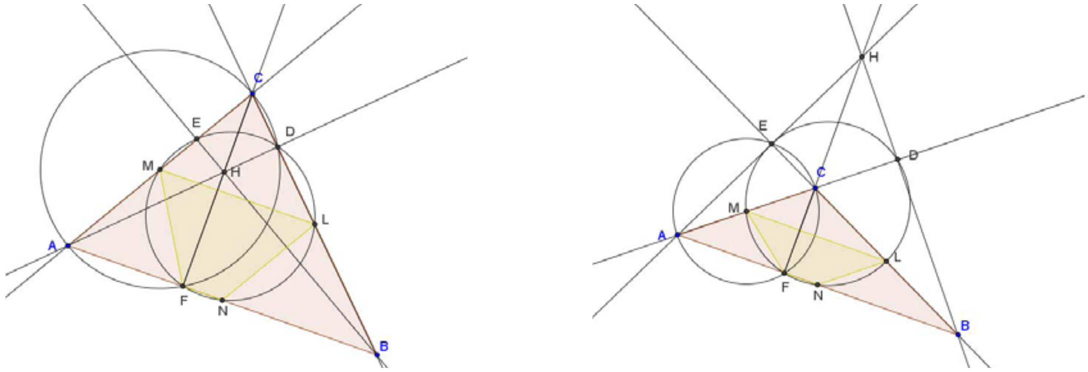


Figure ZZ. The six point theorem: case-invariant proof.

In (Kondratieva, 2011c) we discuss a possible classroom scenario regarding the development of a proof based on Figure ZZ. The cases of acute or obtuse triangle may differ in certain details such as the position of the orthocenter H with respect to the given triangle ABC. At the same time, the fact that MLNF is an isosceles trapezoid and the explanation of this fact survive the transformation from one case to another as shown on Figure ZZ. Again, several BGCs are present in this proof. First, ML is parallel to FN because a midline is parallel to a corresponding side. Second, NL=AC/2 due to another midline-side property. Finally, MF=AC/2 because and AFC is a right triangle with hypotenuse AC, and thus F lies on a circle with diameter AC and center M. We found that it may be useful to discuss one case with the whole class and then ask students to consider the second case on their own paying attention to the details that remain and those that change. It is remarkable that playing with case-invariant solutions-applets students start to notice other geometrical properties which are not employed in the proof of the original statement but could be useful elsewhere. From this particular applet some students noted that if H is the orthocenter in ABC then C is an orthocenter in ABH, the fact that became important for developing their own proof of the Nine Points Circle theorem also studied in the course (see also section 5.5).

4.4. Case-invariant solutions helpful for making mathematical connections.

The last example comes from the fact that was first observed in a DGE and led to the following problem (DeVillier, 2010). Recall that *a parallelo-hexagon* is a hexagon with three pairs of opposite sides being parallel and equal.

Problem 3. *Let ABCDEF be a parallelo-hexagon, and let points G, H, I, J be the midpoints of the sides AB, CD, DE, and FA respectively. The problem is to show that Area(ABCDEF) =2 Area(GHIJ).*

A parallelo-hexagon has several interesting properties. It is a natural generalization of a parallelogram. Each of its main diagonals AD, BE, and CF cuts the parallelo-hexagon in two congruent quadrilaterals. All three main diagonals intersect at one point, call it the center. A parallelo-hexagon remains invariant under rotation around its center by 180 degrees. Justification of these properties along with construction of corresponding applets can be assigned as a preliminary exercise to the students, before they attempt solving problem 3.

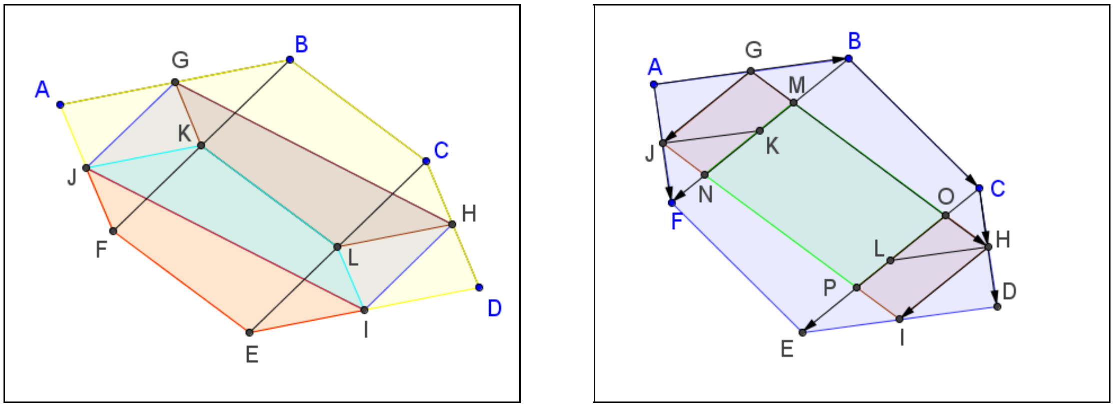


Figure VV. Two approaches to proof that area of GHIJ is half of the area of the parallelo-hexagon.

One possible solution of problem 3 is based on the following area decomposition (See figure VV, left). Let K and L be the midpoints of the diagonals BF and CE respectively. Then GKJ is the middle triangle of ABF, and thus GKJ and GAJ are congruent and their areas are equal. Similarly, we obtain that areas of HLI and HDI are equal.

Now observe that GBCHLK is also a parallelo-hexagon and GH is the diagonal that divides its in two congruent quadrilaterals GBCH and HLKG, which consequently are of equal areas. Similarly, areas of JKLI and IEFJ are equal. Thus we have [GHIJ]=[GKJ]+[GHLK]+[HLI]+[JKLI]=[ABCDEF]/2. Here […] denote the (geometrical) area of corresponding polygon.

It is remarkable that the area relation in Problem 3 remains the same in a non-convex case as well as in the case of self-crossing parallelo-hexagon. This fact can be easily observed by dragging vertices and comparing numerical values of the areas of interest. The problem with presented solution is that it does not survive the transformation across the cases. This particular area decomposition does not illustrate the required area relation and becomes non-informative in the case of self-intersecting hexagon. Thus here we face the situation when we are looking for a case-invariant explanation of the phenomena observed in a DGE.

Let us recall the notion of "algebraic area", or "area with a sign" which allows to preserve information about the orientation of the region's boundary. According to our agreement, the algebraic area of a triangle is the usual geometric area if the triangle is oriented clockwise and the negative of that area if the triangle is oriented counterclockwise. For example, Figure TT (left) shows two congruent triangles of the same geometric area; however the algebraic area of $A_1B_1C_1$ is positive while algebraic area of ACB is negative. Note that this choice of sign is completely arbitrary: things would work just as well if the opposite convention were chosen. The algebraic area of a polygon, broken into a set of triangles oriented according to its boundary orientation, is defined as a sum of algebraic areas of these triangles. Figure TT(right) shows that algebraic area of $A_1B_1C_1D_1$ is the sum of algebraic areas of $A_1B_1C_1$ and $C_1D_1A_1$ each of which is positive. At the same time, algebraic area of a self-crossing quadrilateral ABCD consist of a negative portion ABE' and a positive area E'CD.

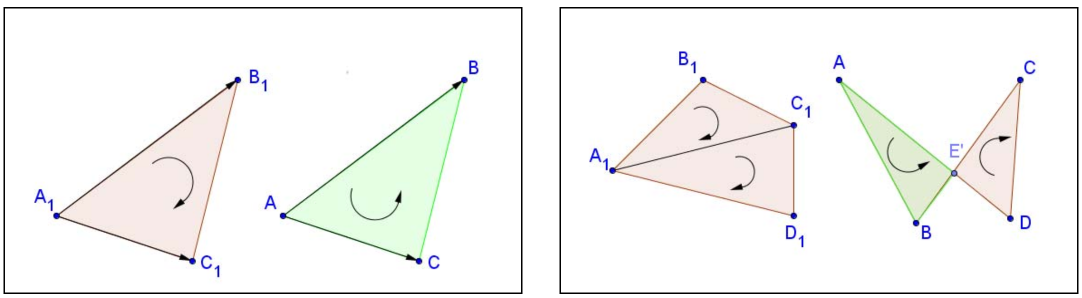


Figure TT. Algebraic area of triangles and quadrilaterals.

With this in hand, we return to the construction of a case-invariant solution of Problem 3. Figure VV (right) suggests that $[ABCDEF]=[ABF]+[BCEF]+[CDE]$. We first look at both the convex case (Figure VV, right)) and the non-convex case (Figure (UU, left). Observe that in both cases $[ABF]=2[GBKJ]=2[GMNJ]$, $[CDE]=2[OHIP]$ and $[BCEF]=2[MOPN]$ since $MN=GJ=BK=BF/2$. Thus we have $[ABCDEF]=2([GMNJ]+[MOPN]+[OHIP])=2[GHIJ]$. Note that all polygons are positively oriented and all algebraic area are positive.

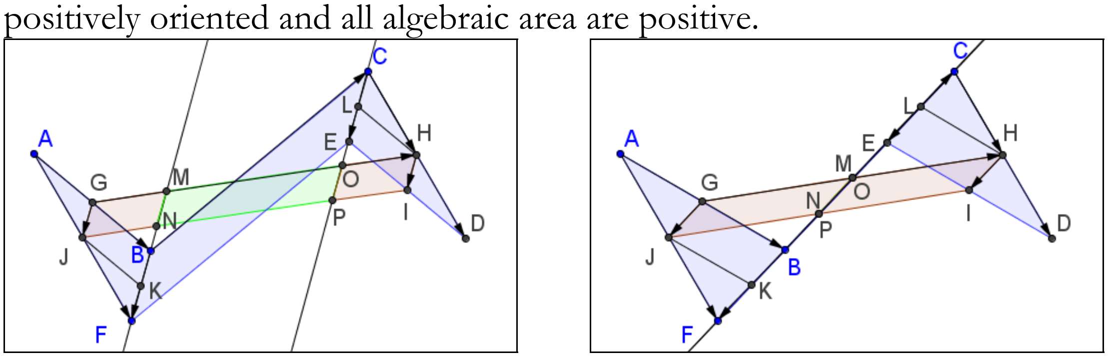


Figure UU. Problem 3: case-invariant geometric solution.

Figure UU (right) illustrates the case when the hexagon degenerates in such a way that points B,C,E,F, and M,O,P,N become collinear and the area

$[BCEF] = [MOPN] = 0$. In this case we have
$[ABCDEF] = [ABF] + [CDE] = 2([GMNJ] + [OHIP]) = 2[GHIJ]$.

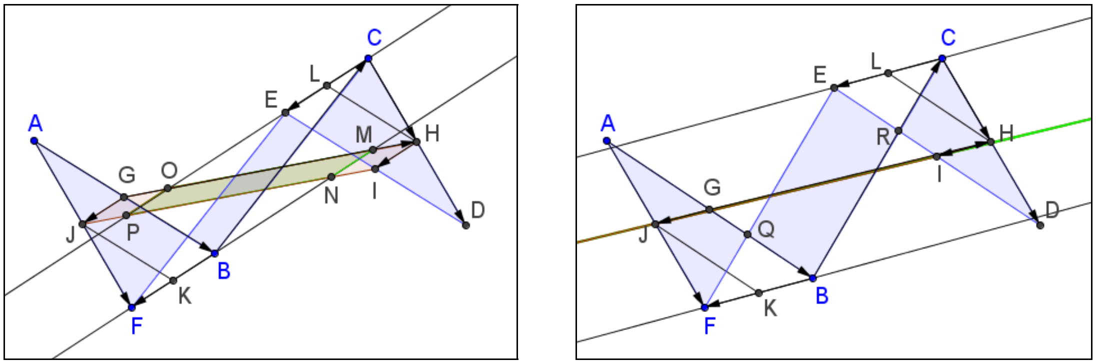


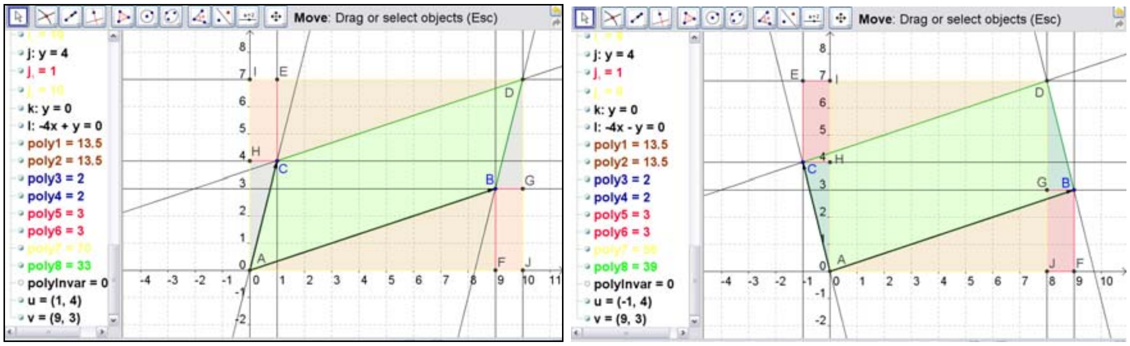
Figure QQ. Problem 3: self-crossing case.

Now, in case of self-crossing hexagon (Figure QQ, left) we have $[ABCDEF] = [ABF] - [BCEF] + [CDE]$. The negative sign results from the fact that BCEF is oriented counterclockwise. One can imagine that rectangle BCEF flips over as we pass to the case of a self-crossing hexagon. Similarly, the parallelogram MOPN is oriented counterclockwise and its area should be subtracted, that is $[GMNJ] - [MOPN] + [OHIP] = [GHIJ]$. Note that relations $[ABF] = 2[GBKJ] = 2[GMNJ]$, $[CDE] = 2[OHIP]$ and $[BCEF] = 2[MOPN]$ still hold in this case. Thus again, $[ABCDEF] = 2[GHIJ]$.

Another special case of problem 3 is presented in Figure QQ (right). Here we see that G,H,I,J are collinear and thus the area [GHIJ]=0. At the same time we see that [ABF]+[CDE]= [BCEF], and BCEF is oriented counterclockwise, so [ABCDEF]=0.

It is important that experimenting with DGE allows us to clarify and confirm the meaning of the relation presented in problem 3. Precisely speaking, this relation refers to algebraic areas in order to be case-invariant. When it comes to the signs of the contributing area-terms, visual representation of a polygon may be ambiguous if the order of vertices is not specified.

The case-invariant solution allows learning and rethinking other branches of mathematics besides synthetic geometry. It is well known that algebraic area of a parallelogram with vertices $A(0,0)$, $B(v_1, v_2)$, $C(u_1, u_2)$ and $D(v_1 + u_1, v_2 + u_2)$ can be calculated as $\{ABDC\} = v_1 u_2 - v_2 u_1$



FigureWW: Computer screen as one works on the *Area of a Parallelogram* problem.

This fact can be proved by area decomposition presented in Figure WW. One needs to distinguish geometrical cases but algebraic expression in terms of the components of vectors $\vec{v} = A\vec{B}$ and $\vec{u} = A\vec{C}$ happens to be case-invariant. For example, for the configuration presented in (Figure WW, left) we have $[ABDC] = [AJDI] - [ABF] - [DEC] - [ACH] - [BGD] - [FJGB] - [HCEI]$. Thus, $[ABDC] = [AJDI] - 2([ABF] + [ACH] + [HCEI])$, and consequently $[ABDC] = (u_1 + v_1)(u_2 + v_2) - v_1v_2 - u_1u_2 - 2u_1v_2 = v_1u_2 - u_1v_2$. However, for the case presented in (Figure WW, right) we have $[ABDC] = [AJDI] - [ABF] - [DEC] + [ACH] + [BGD] + [FJGB] + [HCEI]$. But since component u_1 is now negative, we have relation $\{ABDC\} = v_1u_2 - u_1v_2$ valid in both cases. Note that this expression changes the sign as we interchange vectors u and v, which precisely corresponds to the change of orientation of the parallelogram {ABDC} = - {ACDB}. For this very reason when we talk about geometrical area we must take the absolute value of the corresponding algebraic area.

The expression for the algebraic area of a parallelogram is known in linear algebra as a determinant of a 2X2 matrix with the first row composed of the two coordinates of point B and second row composed of the two coordinates of point C. The idea that determinants are related to areas of parallelograms and volumes of parallelepipeds was successfully employed by mathematicians since the 19^{th} century, but unfortunately many contemporary students of mathematics are not familiar with this fact. In addition, 2X2-determinants define the vector equal to the cross-product of two 3D vectors. In its turn, the length of cross-product $\vec{v} \times \vec{u}$ represents the geometric area of the parallelogram formed by the two 3D-vectors v and u. To account for area orientation one should also consider the direction of the cross product vector. For example, in our case 3D-vectors $\vec{v} = A\vec{B} = (v_1, v_2, 0)$ and $\vec{u} = A\vec{C} = (u_1, u_2, 0)$ form parallelogram ABDC. Then the cross product is found as

$\vec{v}\times\vec{u}=(0,0,v_1u_2-v_2u_1)$ and the algebraic area is equal to the 3rd component of the cross-product $\{ABDC\}=v_1u_2-v_2u_1=(\vec{v}\times\vec{u})_3$.

Students usually study cross-product in a Vector Calculus course. Problem 3 presents a great opportunity to recall the formula along with its geometrical proof as we presented it here, and elaborate on the notion of algebraic area in the context of self crossing figures. Thus, consideration of a case-invariant solution helps to make mathematical connections between vector calculus, linear algebra and Euclidean geometry. The fact that the problem allows a pure geometrical solution (discussed above) as well as linear-algebraic solution, presented below, makes this problem an interconnected one (Kondratieva, 2011b).

The idea of the case-invariant pure geometric solution is supported by the following linear-algebraic consideration. First of all, the definition of a parallelo-hexagon can be re-written as vector equality: $\vec{AB}=\vec{ED}$, $\vec{BC}=\vec{FE}$, $\vec{AF}=\vec{CD}$. Thus we obtain relations $\vec{BF}=\vec{BA}+\vec{AF}=\vec{DE}+\vec{CD}=\vec{CE}$ and $\vec{GH}=\vec{AB}/2+\vec{BC}+\vec{CD}/2=\vec{ED}/2+\vec{FE}+\vec{AF}/2=\vec{JI}$. Consequently, we observe that $GMNJ$ is a parallelogram. Since $\vec{BK}=\vec{BF}/2=\vec{GJ}=\vec{MN}$, we have the following area relations $\{GMNJ\}=\{JBKJ\}=\{ABF\}/2$. The equality of areas of two parallelograms $\{GMNJ\}=\{GBKJ\}$ can also be explained by the Cavalieri's principle since both of them have the altitude and base of the same lengths. Similarly, we obtain that both MOPN and BCEF are parallelograms. Also, since $\vec{BF}/2=\vec{MN}$, we have $\{MOPN\}=\{BCEF\}/2$. Finally, the vector solution can be written as $\{GHIJ\}=(\vec{GH}\times\vec{GJ})_3=((\vec{AB}/2+\vec{BC}+\vec{CD}/2)\times(\vec{BF}/2))_3$. After simplifications and using relations $\{ABF\}=(\vec{AB}\times\vec{BF})_3/2$, $\{BCEF\}=(\vec{BC}\times\vec{BF})_3$, $\{CDE\}=(\vec{CD}\times\vec{CE})_3=(\vec{CD}\times\vec{BF})_3$, we have the required formula $\{GHIJ\}=\{ABF\}/2+\{BCEF\}/2+\{CDE\}/2=\{ABCDEF\}/2$.

This point of view gives area relations which remain unchanged across the cases and hence suggests the partition of the hexagon used above in construction of a case-invariant synthetic solution.

5. Students' responses

Instructors' aim in this project was to familiarize students with a DGE mostly by demonstrating some constructions during the lectures. The assignment included tasks of constructing applets in GG related to theorems and problems assigned. It was of interest to know to what extent the students perceived this approach to teaching the course as being helpful and to observe how the students are going to use dynamic software during their study pertinent to this course.

Thirteen students participated in the study. At the end of the semester the students were asked to respond to a questionnaire. Summary of students' responses is given in the Appendix. Few students volunteered to share their thoughts during semi-structured conversations with the instructor after the study was completed. Students' responses can be united under the following themes.

5. 1. *Assistance required starting and creating applets.*

The opinions were divided between those few who are quick in figuring out new software and the small majority who struggled at the beginning.

"The first time I used GG I thought it was fairly straightforward. When I was unsure of how to create an isosceles triangle I googled how and got a number of results at least one of which was helpful."

"Very easy to use. All my difficulties are easily solvable by a quick search on line. I found the icons and descriptive hints made it easier to understand. What to do."

"At first the program seemed as frustrating as others. But once I started using it more I realized it was not the devil after all."

The degree of challenge experienced by a student while learning to use GG could be a reflection of their previous experiences with computers but also the fact that some students set higher goals for themselves in terms of the quality of their drawings.

"My first experience with GG was confusing. However, after finishing the first assignment I realized that once figuring out the program it is less tedious than drawing by hands. I find it hard to learn software but I like it now that I know it."

Some students at first have not realized that geometrical knowledge is required to draw certain figures and were actually looking for a button that produced desired effect.

"My biggest challenge has been figuring out how to draw certain things that do not have a button. This can be frustrating at times."

Some students' figures especially at the beginning were only static and involved concrete measures of angles and lengths which was not a part of the problem description. Gradually students learn how to produce robust constructions that sustain dragging.

“I have trouble producing figures that do not change their configuration when you click on a point and move it. I still have trouble locating the one third point on a line.”

5.2 Does it call for or help with explanations?

Many students were actually happy to accept visual evidence provided by GG.

“GG is a tool to understand better but it does not generate a need for a proof in my mind. It allows producing an accurate drawing and it is never wrong so I would rather accept the fact I observe than start to doubt and seek for a justification. In general I am happy to accept things without proofs.”

Again, the opinions were divided which apparently reflected a variation in students’ background and attitude for learning.

“I found it convincing to see some properties on the picture. So in these cases it does not call for a proof as such. But it is also interesting to know why things are working. So I would not say that empirical evidence completely takes away the necessity of proving for me.”

Some students reported that the call for explanation could appear in them only if the figure was in some way unexpected.

“I read geometrical statement and always imagine a figure in my mind. Sometimes GG adds to my mental picture and that is when I possibly ask myself “why?”.

But even then interacting with the figure was not always helpful to find such explanations.

“When I see the result I sometimes find it difficult to start from BGC and facts to explain why it is so. GG is helpful to produce nice figures that sometimes made it obvious that BGC or what we are required to prove is true, but I do not find the applets make it easy to produce the solutions [explanations].”

Students agreed that experimenting with applets was not necessarily the best way for them to learn thinking deductively, but nevertheless it was helpful in some other way.

"I took a course in formal logic which helped my development of deductive reasoning. GG does not have the same effect on me, but it allows seeing the details and developing intuition based experimentation with figures, which I actually like."

5.3. Is the dynamic feature helpful? What the dynamic feature is used for?

Making connection between symbolic and visual representation and better understanding of the meaning of the statements were the most popular responses.

"I like the interactive aspect of GG and being able to shift and experiment with figures is extremely helpful with visualizing the problems on the assignments."

"It is helpful when reading the problems, because it is easy to draw figures, and if you do it right the figures can be changed around, yet preset relations will still hold... The biggest challenge is to create relations which hold."

Some students used measurement function and relied on the precision offered by the software. For many resolving some visual paradoxes was easy with GG.

"GG makes drawing less time consuming and more precise. It is good for checking your answer and proof by using the measurement feature."

Many thought that working with applets facilitates exactness of their thinking.

"With GG my thinking was more confident and explicit. I used to rely on visual images in a similar fashion when I studied physics."

Students appreciated the ability to embrace different cases of a problem in one applet.

"It has allowed me to really understand the problem and to believe what I am trying to prove. I like that I can manipulate each figure to see different situations without having to redraw a new figure."

While it was not always explicitly emphasized in the lectures, the students seemed grasping the significance of *construction' case-invariance* and searched for the elements or ideas that remain important in all cases.

"I realized that in order for my figures to have any significance they could not be destructible and had to work in every situation. Now that I know how to do that, the figures are much more "usable".

"I like the ability to generate specific figures that are correct no matter how they are transformed."

Some students associated their understanding of the course with their ability to create the applets and found applets helpful for retention of their knowledge

"Although I could not get the graphs to be exactly right (sometimes I got them to look right), GG helped with my overall understanding. The better I become with the program hopefully it will help me understand more."

"GG for most part displays images that I perceive in my mind. Other times, it helps my understanding of geometric structures via graphical manipulations. Experimenting with figures during my study helped me to recall some statement on the exam."

However other students thought of using GG as an independent yet pleasant activity.

"I do not find GG that helpful towards my understanding of the assignments, but it is my favorite part. The challenge is to make a figure that follows multiple rules and being flexible at once, but this is a challenge I always look forward to solving."

5.4. Students' habits in learning proofs.

Several proofs were discussed during the lectures with different amounts of details. BGCs were emphasized and applets were used to demonstrate various cases. For some proofs the students were asked to read the book and explore statements with their applets first, and then the proofs were discussed with the whole group if necessary. Many students found this approach helpful.

"I prefer a combination of reading and explanation of teacher. I like both to hear explanations after reading and read after in-class discussions. The point is that you get a couple of slightly different perspectives."

When constructing proofs, some students had more problems with “translating words into applets” than re-drawing applets with given properties posted on-line

“I like a combination of given explanations and experimentations. I like to create applets: it is challenging but also interesting. I would rather redraw an applet with given relations than create my own applet by transforming word problems onto images on the screen.”

Few students preferred to construct their own proofs before or instead of reading the book which was actually encouraged by the instructor.

“It is my general habit to start doing homework in the evening and finish up in the morning. Sometimes ideas come to me overnight. So I experiment with GG in the evening and finish my proof on paper next day. I like to create my own proof from scratch before reading the book. I draw figure first to see what the problem or theorem is saying. Then I play with the figure trying to find a solution. I found proofs in the book sometimes long and challenging to understand.”

5.5. Aha moments.

Several students reported having an instant insight while trying to create their own proofs. It usually happened after working on a problem and then either having a break, or browsing through lecture notes, talking to a peer or the instructor, or changing the strategy, e.g. working backward.

“I had been working on one question for about 2 hour. I took a break but upon returning to my problem I was still stuck. I went to talk the instructor and explained my ideas. All she said was “these angles sum to 180” and it was an instant aha moment. I love these moments, everything just clicks and it is a wonderful feeling.”

For very few students the insights happen while using GG applets so that they could relate it to the manipulation of the figure. Using *Trace* function was found to be helpful. The following episode refers to Problem 2 from section 4.2.

“… one problem asked for a locus of points and I could not imagine it in my mind. Then I made an applet and observed the locus which happened to be a circle. But then I almost immediately saw lines intersecting at right angle and this was an explanation. After this insight I had to sit down and write the proof in details.”

Most of the students agreed that "with GG I could see certain configuration clearer which sometimes generated an aha-moment in terms of better understanding of what the textbook was saying."

While working with applets some students reported observing "extraneous" facts, that later turned to be useful for other proofs. The following episode refers to Figure ZZ (right) originally used in the proof of the Six point circle theorem.

"I accidentally found a proof of the Nine point circle theorem. I looked at picture of an obtuse triangle ACB (C >90deg) and its altitudes and the orthocenter H. But I thought of triangle AHB and C as its orthocenter. First I thought it was my mistake, but then I checked and found that this point of view is also possible and is actually quite useful for the proof of the theorem."

Conclusions

This paper attempted to conceptualize and describe an innovative teaching approach in synthetic Euclidean geometry at the undergraduate university level. Freedom of exploration was given to the students within the frames of traditional curriculum shaped by a book and the instruction. The explorations were supported by a DGE that was also used during the in-class discussions. A particular emphasis was made on learning and recognition of basic geometric configuration in both static and dynamic solutions. Case invariance of several solutions was demonstrated as one of the keys to understanding, knowledge connection and retention.

Based on the instructor's observation and students' responses, which were collected in order to inform the next course delivery, the following conclusions can be reported.

First, students need a bit more directions regarding how to use DGE and, in particular, how to construct applets with required properties. Students have to be explicitly oriented to the fact that they need to use their geometrical knowledge in order to create "indestructible by dragging" constructions, and this type of problems should be given more emphasis during the classroom discussions.

Second, in order to strengthen the descriptive-pictorial relations, students may be assigned problems of conversion of an observed applet into a word statement before they are asked to build applets related to word problems. Same as with static BGS, students should be asked to protocol the properties they observe when interacting with the applets implementing BGC. Interacting with applets will help the students to notice certain properties and move from concrete to conceptual level. The concepts formulated at first in students' own naïve language should then be converted in to formal symbolic statements acceptable by the mathematics community. The book and lectures should set an example of the latter.

Third, reading and analysis of formal proofs from a textbook accompanied by experimentation in a DGE should be continued. Perhaps, more transparency (Hemmi, 2008) regarding the structure and logic of the proofs needs to be introduced. Case-invariance of proofs, when it takes place, should be explicitly emphasised as well as its possible roles in getting general ideas from particular cases and making mathematical connections. The fact that synthetic geometry solutions can be supported by ideas of, and often restated in the language of linear algebra, analytic geometry or complex analysis, should be discussed in more depth in a variety of examples.

In my view, this pedagogical experiment of introduction of a DGE at the university level Euclidean geometry course confirmed many important points regarding the use of technology in teaching that are found in the literature. The freedom of experimentation offered by the use of DGE needs to be very well structured by the instructor in order to help students to conceptualize geometrical knowledge at both intuitive and formal deductive levels. Our next course delivery informed by the observations reporter in this article will hopefully shed more light on the subject.

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Appendix: Students survey on the use of Geogebra in the course
Background

I have a solid background in geometry from my grade (k-12) school	Yes=5	No=8
I did many proofs in my grade school or university courses	Yes=5	No=8
I am learning a lot of new things about geometry in this course	Yes=13	no
I am developing my understanding of and ability to prove to a higher degree in this course	Yes=12	No=1
In general I like the use of technology to assist my learning	Yes=7	No=6
I have no problem using Geogebra as a tool for drawing (static) pictures	Yes=8	No=5
I have no problems in using Geogebra as a tool for creating (dynamic) applets	Yes=8	No=5

Preferences

I like to read proofs given in the book and fill in possible gaps in them	Yes=5	No=8
I like to create my own proofs from scratch	Yes=7	No=6
I like to get some directions in class and complete the same problems at home	Yes=12	No=1
I like to experiment with applets made by others	Yes=4	No=9
I like to illustrate statement and proofs with my own applets made in Geogebra	Yes=7	No=6
I like to create my own applets similar to (or better than) ones made by others	Yes=8	No=5
I like to resolve fallacies and geometrical paradoxes	Yes=7	No=6

The use of GeoGebra helps me

To interpret and understand theorems and statements in geometry	Yes=11	No=2
To make a connection of symbolic and visual representations	Yes=12	No=1
To construct mathematical strategies and ideas	Yes=8	No=5
To test my ideas and adopt an action-oriented way of thinking	Yes=8	No=5
To develop reasoning skills and the notion of proof	Yes=6	No=7

I found that creating applets in GeoGebra is

Aesthetically pleasant and rewording	Yes=11	No=2
Time consuming	Yes=12	No=1
Insightful	Yes=12	No=1
Helping to activate my knowledge	Yes=11	No=2

Geogebra also

encourages me to make and test conjectures,	Yes=11	No=2
encourages to move from naïve to logical thinking;	Yes=9	No=4
gives immediate feedback on my actions	Yes=9	No=4
facilitates exactness of my mathematical thinking;	Yes=11	No=2
allows me to search for geometrical relationship that may seem beyond my grasp at that moment	Yes=8	No=5
allows me to try a larger range of possibilities compare to pen and paper approach	Yes=13	No=0