Problem 37

Let $ABCD$ be a convex quadrilateral. Find a necessary and sufficient condition for a point P to exist inside $ABCD$ such that the four triangles ABP, BCP, CDP, DAP all have the same area.

Answer. A necessary and sufficient condition is for one of the diagonals AC and BD to bisect the other.

Solution 1. To prove sufficiency, let Q be the intersection of AC and BD. Without loss of generality, we may assume that $AQ = CQ$. Let P be the midpoint of BD. Furthermore, let R be the foot of the perpendicular from A to BD and let S be the foot of the perpendicular from C to BD.

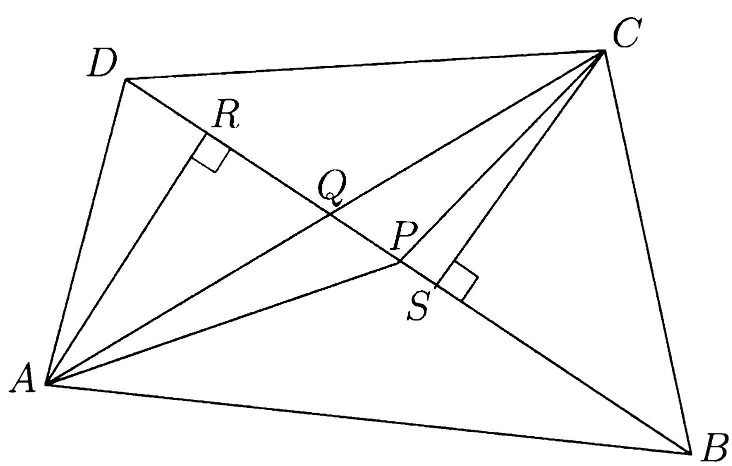

FIGURE 9

Since $BP = DP$ and since AR is the altitude for triangles ABP and DAP, these triangles have equal area. Similarly, the areas of triangles BCP and CDP are equal. Triangles ARQ and CSQ are congruent; therefore, $AR = CS$ and, hence, the areas of triangles ABP and BCP are equal. We conclude that the four triangles ABP, BCP, CDP, and DAP all have the same area.

To prove the necessity, assume that P is a point in the interior of $ABCD$ for which triangles ABP, BCP, CDP, and DAP have the same area (see Figure 10). Since triangles ABP and BCP have the same area, vertices A and C are equidistant from line BP. Because the quadrilateral $ABCD$ is convex, A and C lie on opposite sides of the line BP. An argument using congruent triangles (similar to the one in the sufficiency proof above) shows that the line

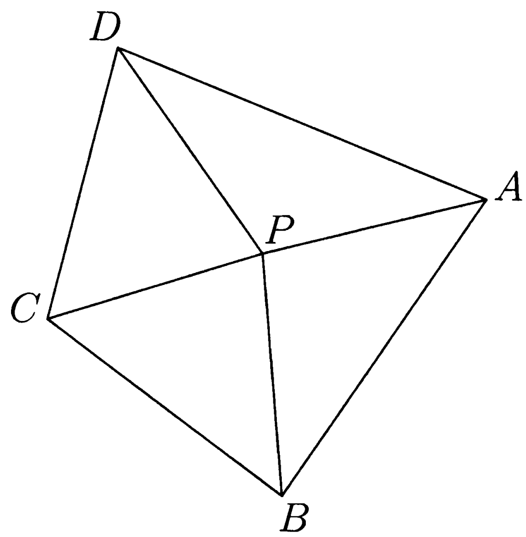

FIGURE 10

BP must actually pass through the midpoint of the line segment AC. Similarly, the line DP must also pass through the midpoint of AC. If B, P, and D are not collinear, then P must be the midpoint of AC. But then, since BCP and CDP have the same area, B and D are equidistant from the line AC and thus AC bisects the segment BD. If B, P, and D are collinear, then P must be the midpoint of BD. In this case, A and C are equidistant from the line BD, hence BD bisects the segment AC, completing the proof.

Solution 2. We prove the sufficiency as above. To prove the necessity, let $\theta_1, \theta_2, \theta_3$, and θ_4 be the measures of the angles APB, BPC, CPD, and DPA, respectively (see Figure 11). The area of an arbitrary triangle is half the product of two sides and the sine of their included angle. The product of the areas of triangles APB and CPD equals the product of the areas of BPC and DPA. Thus,

$$\sin\theta_1 \sin\theta_3 = \sin\theta_2 \sin\theta_4,$$

or, from the formula for the cosine of a sum,

$$\cos(\theta_1 - \theta_3) - \cos(\theta_1 + \theta_3) = \cos(\theta_2 - \theta_4) - \cos(\theta_2 + \theta_4).$$

Since $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi$, this implies

$$\cos(\theta_1 - \theta_3) = \cos(\theta_2 - \theta_4).$$

There is no loss of generality in assuming

$$\theta_3 \leq \theta_1 < \pi \quad \text{and} \quad \theta_4 \leq \theta_2 < \pi$$

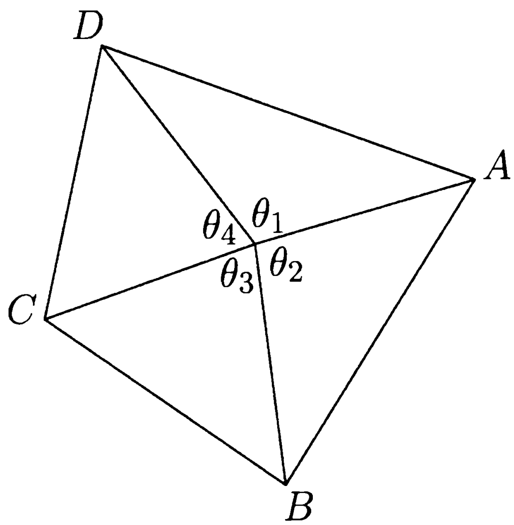

FIGURE 11

(if necessary, rename the vertices and/or reflect the quadrilateral). Under these conditions, $\theta_1 - \theta_3 = \theta_2 - \theta_4$, hence $\theta_1 + \theta_4 = \theta_2 + \theta_3$, from which we conclude that B, P, and D are collinear. As in the first solution, this implies BD bisects AC.

Problem 38

Suppose that n dancers, n even, are arranged in a circle so that partners are directly opposite each other. During the dance, two dancers who are next to each other change places while all others stay in the same place; this is repeated with different pairs of adjacent dancers until, in the ending position, the two dancers in each couple are once again opposite each other, but in the opposite of the starting position (that is, every dancer is halfway around the circle from her/his original position). What is the least number of interchanges (of two adjacent dancers) necessary to do this?

Solution. The least number of interchanges required is $n^2/4$.

If the dancers are numbered $1, 2, \dots, n$ as we move clockwise around the circle, first begin by interchanging dancer number $n/2$ with dancer $n/2+1$, then with dancer $n/2 + 2, \dots$, and finally with dancer n. We have made $n/2$ interchanges and dancer $n/2$ is in the correct position. Next, interchange dancer $n/2 - 1$ with dancers $n/2 + 1, n/2 + 2, \dots, n$. Continuing this process until dancer 1 has switched with dancers $n/2+1, n/2+2, \dots, n$, we arrive at the de-