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(Unpublished classroom notes)

CONSTRUCTION AND INVESTIGATIONS OF GOLDEN TRAPEZOIDS

The Golden Ratio, which is one of the greatest treasures of mathematics, was discovered by Pythagoras of Samos (c 585-c501 B.C.) and his students by exploring irrational numbers. Leonardo of Pisa (Fibonacci) was the first person to generate the ratio from specific numbers. The Fibonacci ratio converge to the Golden Ratio ($\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$). The Golden Ratio has also been investigated in the field of geometry. The ratio has a long history, and many excellent references are available [**1, 3, 6**]. It is a centerpiece in the Walt Disney classic film *Donald Duck in Mathemagic Land*. Johann Kepler (1571-1630) was reported to have considered geometry as having two great treasures: one is the theorem of Pythagoras; the other is the division of a line into extreme and mean ratio, which he called "the divine proportion." The first he compared to a measure of gold; the second he named a precious jewel. Examples of current work in this area may be found in [**2, 4**]. Triangles and rectangles whose sides are in the proportion of the Golden Ratio are known as Golden Triangles and Golden Rectangles respectively. In the past, investigations of other figures have been limited by traditional compass and straight edge construction methods. By taking advantage of current technology, I have been able to investigate additional figures whose sides are in the Golden Ratio. I call these shapes Golden Trapezoids. Their construction, properties and related proofs will be the focus of this paper.

In most cases, the use of technology has been justified as a tool that enables one to make investigations faster than using traditional methods. The dynamic nature of technology allows us to change parameters and view several forms of a particular phenomena easily and quickly. In the construction of the Golden Trapezoids, however, the use of technology appears to be indispensable. Difficult and heretofore perhaps unknown straight edge and compass methods to construct Golden Trapezoids have been uncovered by investigating the properties of these figures in a dynamic environment. The processes discussed in this paper help undergraduates to appreciate the power of technology and the value in mathematical investigations.

THE GOLDEN-O TRAPEZOID

I define a Golden-O Trapezoid as a trapezoid with the ratio of both pairs of opposite sides equal to the Golden Ratio. To construct one dynamically using application software such as Geometer's Sketchpad [**5**], start as in figure 1, with a triangle ABM with $\frac{|\overline{MA}|}{|\overline{MB}|} = \phi$. The point D is moved along $\overline{AM}$ until $\frac{|\overline{AB}|}{|\overline{CD}|} = \phi$.

The trapezoid ABCD is a Golden-O Trapezoid.

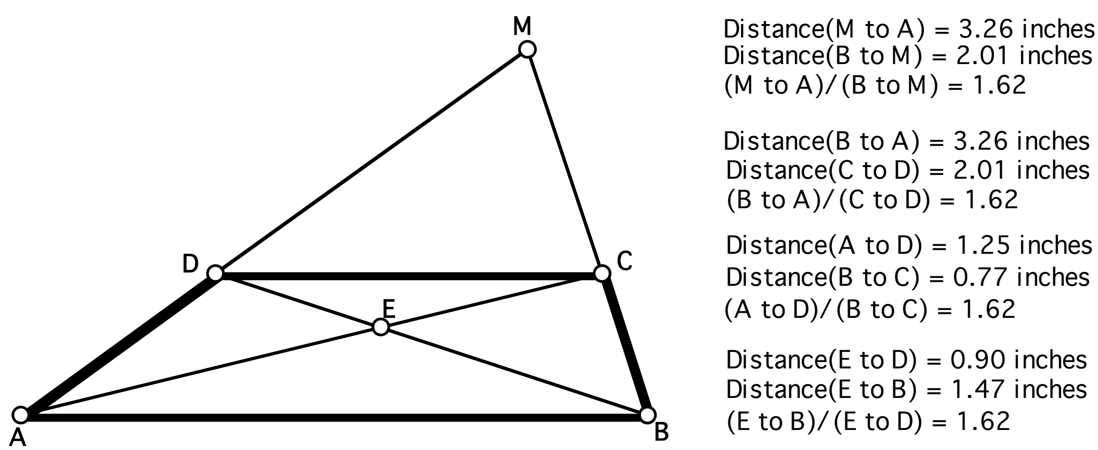

Figure 1. Golden-O Trapezoid

We may make conjectures about the diagonals of trapezoid ABCD (Figure 1). Do they always intersect each other in the Golden Ratio? Further investigation by measurement indicates this is the case. We note that triangles BEA and DEC are similar and $\frac{|\overline{AB}|}{|\overline{CD}|} = \phi$ Hence, $\frac{|\overline{BE}|}{|\overline{DE}|} = \frac{|\overline{AE}|}{|\overline{CE}|} = \phi$.

Can we use traditional methods to construct the Golden-O Trapezoid? Take any segment AM. Construct a point D on the segment to divide AM in the golden ratio. There is a standard construction for the mean and extreme ratios, which does not require description in this article. For example, there are scripts available with the Geometers Sketchpad [**5**] that will do the construction for you. Now, along some line other than through M construct $\overline{MB} = \overline{MD}$ and draw $\overline{AB}$ to complete triangle ABM (Figure 2). Construct CD parallel to AB. From basic geometry, a segment parallel to the base of the triangle cuts the sides in equal ratios, cuts off segments of the same ratio, and is itself in the same ratio to the base of the triangle. Therefore, $\frac{|\overline{MA}|}{|\overline{MD}|} = \frac{|\overline{MC}|}{|\overline{CB}|} = \frac{|\overline{AB}|}{|\overline{CD}|} = \frac{|\overline{AD}|}{|\overline{BC}|} = \phi$. ABCD ia a Golden-O-trapezoid.

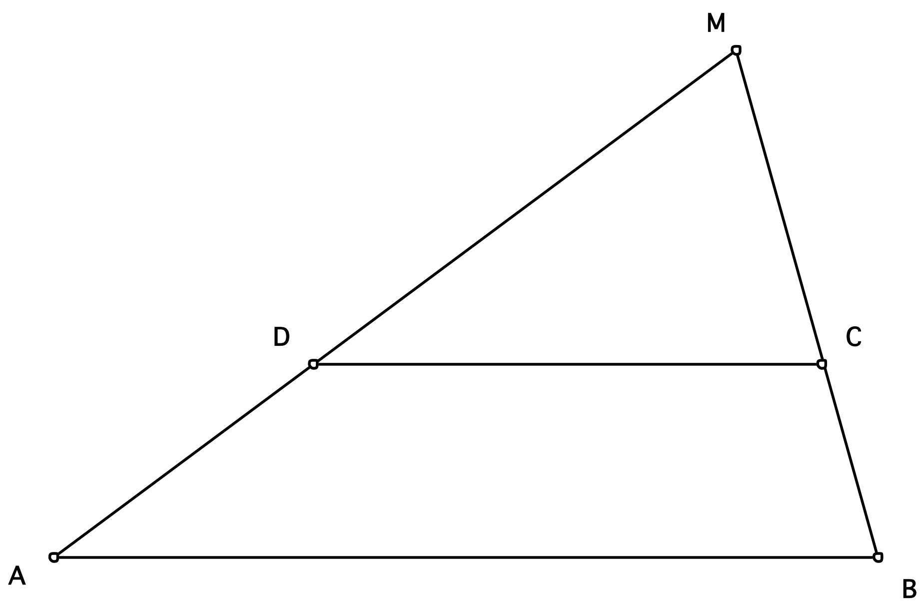

Figure 2. Construction of Golden-O Trapezoid

Now construct BD and AC to intersect at E as shown in Figure 3. The areas of triangles ABD and ABC are equal since they have the same base and are between the same parallel lines. Similarly, the areas of triangles CDB and CDA are equal. However, the areas of triangles BDC and BDA do not seem to be equal. We might conjecture that the ratio of the areas of these triangles are in the Golden Ratio. By using the measurement tool in Geometer's Sketchpad, we will in fact notice that this is true. Both triangles have a common base $\overline{BD}$. Since $\overline{AB}$ and $\overline{CD}$ are in the Golden Ratio, the heights of the triangles will also be in the Golden Ratio. Therefore the ratio of the areas of the triangles equals the Golden Ratio.

If lines are constructed through points D and C parallel to the diagonals of the trapezoid ABCD, we obtain a Golden-O-trapezoid DCC_1D_1 with intersection of diagonals at E_1. Successive constructions as shown indicate that the points EE_1E_2... are colinear with M and E_1 divides E and E_2 in the Golden Ratio. Also figures $D_1C_1C_2D_2$, D_2C_2..., ... are all Golden-O-trapezoids.

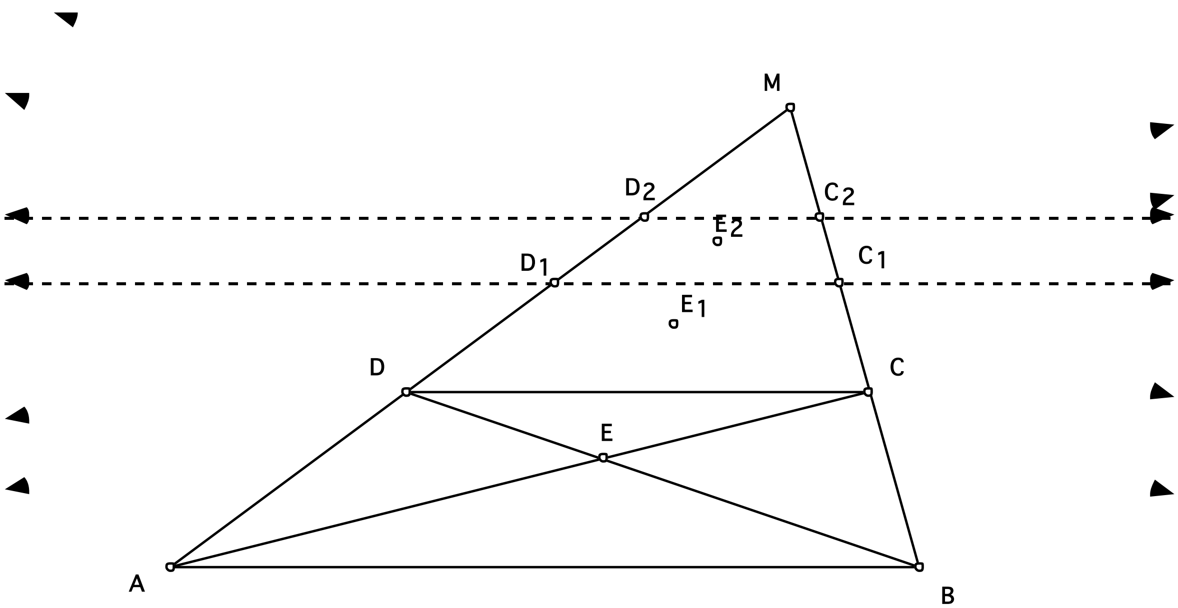

Figure 3. Successive constructions of Golden-O-trapezoids

THE GOLDEN-A TRAPEZOID

Let us now consider trapezoids in which two pairs of adjacent sides have the same ratio. The trapezoid AECD satisfies this condition (figure 4). It is created by constructing parallelogram ABCD with D on the circle with center A.

We can extend $\overline{AB}$ to point F, and construct a point E which moves freely on $\overline{AF}$. We position E along $\overline{AB}$until the ratio $\frac{|\overline{AE}|}{|\overline{EC}|} = \frac{|\overline{DC}|}{|\overline{AD}|}$. We construct the parallelogram ABCD such that $\frac{|\overline{DC}|}{|\overline{AD}|} = \phi$. In this special case, if we position E along $\overline{AF}$ such that $\frac{|\overline{AE}|}{|\overline{EC}|} = \phi$, then I define the trapezoid AECD as a Golden-A Trapezoid.

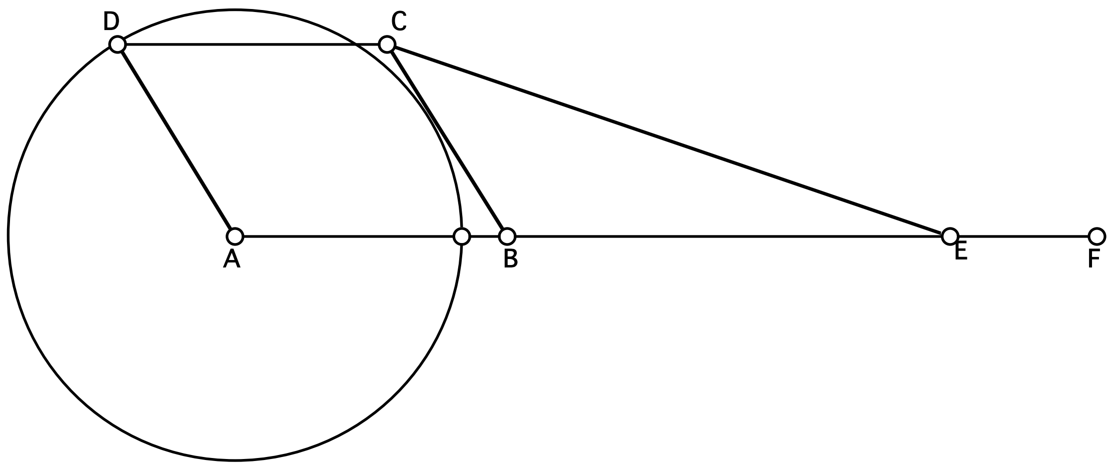

Figure 4. Golden-A Trapezoid

By inspection, we can make conjectures about the diagonals of the Golden-A Trapezoid. The diagonals do not seem to intersect each other in the Golden Ratio. If the diagonals intersect each other in the Golden Ratio, then the parallel sides will be in the Golden Ratio, which is not the case. Is there a trapezoid which is both Golden-O and Golden-A?

THE SUPER GOLDEN TRAPEZOID

The climax of these constructions is the special case when both ratios of pairs of the opposite sides, as well as the ratio of the two pairs of the adjacent sides of a trapezoid, are the Golden Ratio. We construct a Parallelogram ABCD (figure 5) such that D moves freely on the circle with center A, and point E moves freely on $\overline{AB}$extended. Position D and E such that, in trapezoid AECD,

$\frac{|\overline{DC}|}{|\overline{AD}|} = \frac{|\overline{EA}|}{|\overline{CE}|} = \frac{|\overline{EA}|}{|\overline{DC}|} = \frac{|\overline{CE}|}{|\overline{AD}|} = \phi$. A trapezoid with these features is called a Super Golden Trapezoid.

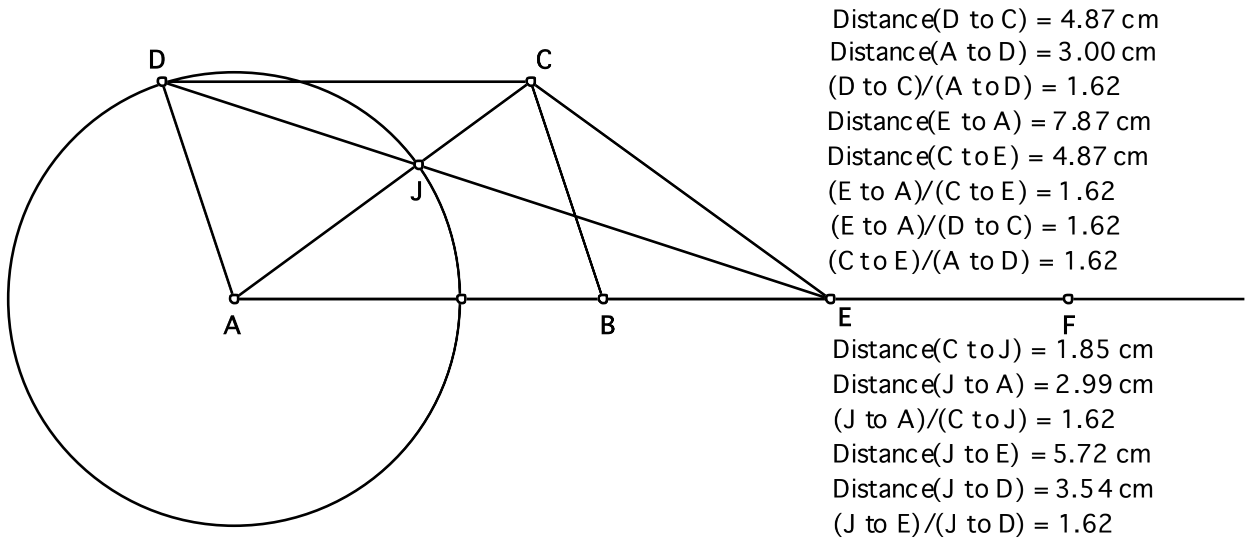

Figure 5. Super Golden Trapezoid

If such a trapezoid can be drawn, then since $\frac{|\overline{AE}|}{|\overline{DC}|} = \frac{|\overline{AE}|}{|\overline{CE}|} = \phi, \overline{DC} = \overline{CE}$. If $\overline{AD} = 1$, and $|\overline{DC}| = \phi$, then $|\overline{AE}| = \phi^2$ because $|\overline{AE}| = \phi \bullet |\overline{DC}| = \phi \bullet \phi = \phi^2 = \phi + 1$. E has to be both a distant of ϕ from point C and a distant of 1 from the point B, and on the line $\overline{AB}$ extended. Choose E a distant of 1 from B, and measure $|\overline{CE}|$.

Traditional methods for the construction of the Super Golden Trapezoid can now be developed. We construct a Parallelogram with $\overline{AD} = 1$, and $|\overline{DC}| = \phi$. We construct point E given that $|\overline{AE}| = 1 + \phi$. We construct a circle with center E and radius equal to ϕ. We construct a circle with center B and radius 1. The points of intersection of these two circles determine the point C. We construct a line through C parallel to $\overline{AE}$ which intersects the circle with center A and radius 1. One of the points of intersection determines the point D. By first exploring the properties of the Super Golden Trapezoid using the dynamic features of

Geometer's Sketchpad, we have been able to provide a theoretical basis for the traditional methods of its construction.

SUMMARY

There are two types of Golden Trapezoids: the Golden-O Trapezoid and the Golden-A Trapezoid. The "O" in the Golden-O Trapezoid indicates both pairs of the opposite sides of the trapezoid are in the Golden Ratio, while the "A" in the Golden-A Trapezoid indicates the ratio of the longest of the parallel sides and the longest of the legs, and also the ratio of the shortest of the parallel sides and the shortest of the legs, is the Golden Ratio. The Super Golden Trapezoid is both a Golden-O and Golden-A Trapezoid.

The idea (Golden Ratio) is old, the situations (trapezoids) are old, but never have the two been thought about together. While the constructions are non-routine, they provide opportunities to work in familiar situations. Some of the figures we have constructed here represent a new class of Golden Figures. Whereas the Golden Triangle (which is isosceles) and the Golden Rectangle are all symmetric, a Super Golden Trapezoid is certainly not symmetric. The Super Golden Trapezoid has more Golden proportions than the Golden Rectangle and Golden Triangle combined. This could be the beginning of new investigations of even more areas associated with the Golden Ratio.

REFERENCES

1. T. H. Garland, *Fascinating Fibonaccis: Mystery and Magic of Numbers.* Dale Seymour Publications, Palo Alto, 1987.
2. R. Herz-Fischler, A "very pleasant theorem", *The College Mathematics Journal* 24 (1993) 318-324.
3. H. E. Huntley, *The Divine Proportion: A Study in Mathematical Beauty*. Dover Publications, New York, 1970.

4. G. Markowsky, Misconceptions about the Golden Ratio, *The College Mathematics Journal* 23 (1992) 2-19.

5. N. Jackiw, *The Geometer's Sketchpad [Computer Program]*. Key Curriculum Press, Berkeley, 1991.

6. S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*. Ellis Horwood Limited, Chichester, 1989.