Now, since the parallelogram *DABE* is equal to the parallelogram *LABH* (for they are on the same base *AB* and in the same parallels *AB*, *DH*),

and likewise *LABH* is equal to *LAKN* (for they are on the same base *LA* and in the same parallels *LA*, *HK*),

the parallelogram *DABE* is equal to the parallelogram *LAKN*.

For the same reason,

the parallelogram *BGFC* is equal to the parallelogram *NKCM*.

Therefore the sum of the parallelograms *DABE*, *BGFC* is equal to the parallelogram *LACM*, that is, to the parallelogram which is contained by *AC*, *HB* in an angle *LAC* which is equal to the sum of the angles *BAC*, *BHD*.

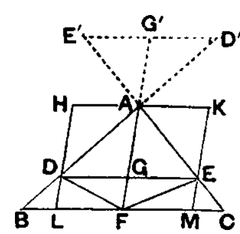
"And this is far more general than what is proved in the Elements about squares in the case of right-angled (triangles)."

Heron's proof that AL, BK, CF in Euclid's figure meet in a point.

The final words of Proclus' note on i. 47 are historically interesting. He says: "The demonstration by the writer of the Elements being clear, I consider that it is unnecessary to add anything further, and that we may be satisfied with what has been written, since in fact those who have added anything more, like Pappus and Heron, were obliged to draw upon what is proved in the sixth Book, for no really useful object." These words cannot of course refer to the extension of i. 47 given by Pappus; but the key to them, so far as Heron is concerned, is to be found in the commentary of an-Nairīzī on i. 47, wherein he gives Heron's proof that the lines *AL*, *FC*, *BK* in Euclid's figure meet in a point. Heron proved this by means of three lemmas which would most naturally be proved from the principle of similitude as laid down in Book VI., but which Heron, as a *tour de force*, proved on the principles of Book I. only. The *first* lemma is to the following effect.

If, in a triangle ABC, DE *be drawn parallel to the base* BC, *and if* AF *be drawn from the vertex* A *to the middle point* F *of* BC, *then* AF *will also bisect* DE.

This is proved by drawing *HK* through *A* parallel to *DE* or *BC*, and *HDL*, *KEM* through *D*, *E* respectively parallel to *AGF*, and lastly joining *DF*, *EF*.



Then the triangles *ABF*, *AFC* are equal (being on equal bases), and the triangles *DBF*, *EFC* are also equal (being on equal bases and between the same parallels).

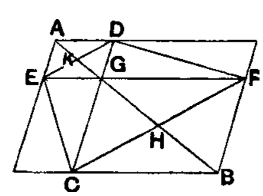
Therefore, by subtraction, the triangles *ADF*, *AEF* are equal, and hence the parallelograms *AL*, *AM* are equal.

These parallelograms are between the same parallels *LM*, *HK*; therefore *LF*, *FM* are equal, whence *DG*, *GE* are also equal.

The *second* lemma is an extension of this to the case where *DE* meets *BA*, *CA* produced beyond *A*.

The *third* lemma proves the converse of Euclid i. 43, that, *If a parallelogram* AB *is cut into four others* ADGE, DF, FGCB, CE, *so that* DF, CE *are equal, the common vertex* G *will be on the diagonal* AB.

Heron produces *AG* till it meets *CF* in *H*. Then, if we join *HB*, we have to prove that *AHB* is one straight line. The proof is as follows. Since the areas *DF*, *EC* are equal, the triangles *DGF*, *ECG* are equal.



If we add to each the triangle *GCF*,

the triangles *ECF*, *DCF* are equal;

therefore *ED*, *CF* are parallel.

Now it follows from i. 34, 29 and 26 that the triangles *AKE*, *GKD* are equal in all respects;

therefore *EK* is equal to *KD*.

Hence, by the second lemma,

CH is equal to *HF*.

Therefore, in the triangles *FHB*, *CHG*,
the two sides *BF*, *FH* are equal to the two sides *GC*, *CH*,

and the angle *BFH* is equal to the angle *GCH*;

hence the triangles are equal in all respects,
and the angle *BHF* is equal to the angle *GHC*.

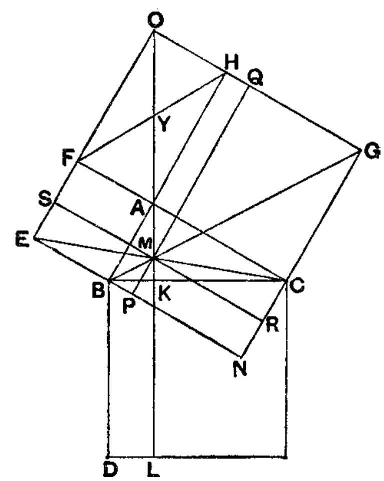
Adding to each the angle *GHF*, we find that the angles *BHF*, *FHG* are equal to the angles *CHG*, *GHF*,

and therefore to two right angles.

Therefore *AHB* is a straight line.

Heron now proceeds to prove the proposition that, in the accompanying figure, if *AKL* perpendicular to *BC* meet
EC in *M*, and if *BM*, *MG* be joined,

BM, *MG* are in one straight line.



Parallelograms are completed as shown in the figure, and the diagonals *OA*, *FH* of the parallelogram *FH* are drawn.

Then the triangles *FAH*, *BAC* are clearly equal in all respects;

therefore the angle *HFA* is equal to the angle *ABC*, and therefore to the angle *CAK* (since *AK* is perpendicular to *BC*).

But, the diagonals of the rectangle *FH* cutting one another in *Y*,

FY is equal to *YA*,

and the angle *HFA* is equal to the angle *OAF*.

Therefore the angles *OAF*, *CAK* are equal, and accordingly

OA, *AK* are in a straight line.

Hence *OM* is the diagonal of *SQ*;

therefore *AS* is equal to *AQ*,

and, if we add *AM* to each,

FM is equal to *MH*.

But, since *EC* is the diagonal of the parallelogram *FN*,

FM is equal to *MN*.

Therefore *MH* is equal to *MN*;
and, by the third lemma, *BM*, *MG* are in a straight line.