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ENRICHMENT FOR THE GIFTED: GENERALIZING SOME GEOMETRICAL THEOREMS & OBJECTS

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This paper briefly discusses the importance of generalization in mathematics, and then presents two possible examples of appropriate generalizations to engage gifted secondary school learners in. The one is the generalization of a familiar theorem for cyclic quadrilaterals to cyclic polygons while the other is the generalization, through the process of constructive defining, of the concept of a rectangle to a hexagon, octagon, etc.

INTRODUCTION

Generalization undoubtedly is a very important mathematical process that dates back to earliest times, and has in modern times become increasingly more valuable within mathematics. The following sample of quotes indicate that generalizing is very fundamental to mathematical thinking and not only expands knowledge, but often contributes to increasing our understanding, for example, by noticing relationships between previously disconnected mathematical concepts, theorems or topics.

> "*Generalization by condensing compresses into one concept of wide scope several ideas which appeared widely scattered before. Thus, the Theory of Groups reduces to a common expression ideas which were dispersed before in Algebra, Theory of Numbers, Analysis, Geometry, Crystallography, and other domains*." – Polya (1954: 30)

> "*Generalising is at the heart of mathematics*." – Johnston-Wilder & Mason (2005, p. 93)

> "*Abstraction and generality go hand in hand ...the issue is no longer whether abstraction (and generalization) is useful or necessary: abstract methods have proved their worth by making it possible to solve numerous longstanding problems such as Fermat's Last Theorem. And what seemed little more than formal game-playing yesterday may turn out to be a vital scientific or commercial tool tomorrow*."
>
> - Ian Stewart (2008, p. 264)

Most formal school systems around the world unfortunately still focus largely on 'routine' problems and the 'basic drill and exercise' of mathematical techniques as critiqued in Sheffield (1999), Lockhart (2002), Mann (2006), Bailey et al (2016), which leaves little opportunity to stretch and challenge gifted learners. Given the pivotal role generalization plays in mathematics as briefly mentioned above, it seems educationally necessary to design suitable activities in mathematics clubs, or other avenues like the internet, for engaging talented mathematical learners (as well as prospective mathematics teachers) in the process of generalization that goes beyond the normal curriculum, and can possibly stimulate them to explore generalizations of their own.

Mathematical creativity for this paper is viewed rather broadly as the discovery or invention of something new to the learner in accordance with Sriraman (2004), not necessarily something completely original. My own teaching and research approach can perhaps be described as that of a 'thought experiment' much in the style of Einstein's thought experiments in physics. These thought experiments have been called 'reinvention' by Freudenthal (1973) and follows what has been called a 'reconstructive' approach by Human (1978:20), namely focusing on exposing learners to, or engaging them with, the 'genuine' mathematical processes by which new content in mathematics is generally discovered, invented and organized. Such design experiments are informed by my own personal experiences of (some elementary) mathematical research as well as major problem-solving and problem-posing works like that of Polya (1945) and many others, as well as my understanding of the history, nature and philosophy of mathematics. A constructivist perspective on the nature of learning, and evaluative feedback from the implementation of these thought experiments, assist in the constant redesigning and adaption of these learning activities.

EXAMPLES OF GENERALIZATION

Following Polya (1954, p. 12) we can define generalization as the movement from a given set of objects to that of a larger set, containing the given one. Therefore, generalization occurs when we move from the observation of a few finite cases to an infinite set, for example, experimentally finding by construction by hand or dynamic geometry software in SOME cases that the perpendicular bisectors of the sides of a triangle are concurrent, we GENERALIZE to the conjecture that it is true for ALL triangles. We could call this inductive generalization since we are moving from the specific to the more inclusive general.

In mathematics, we can also generalize from one infinite class to a larger infinite class of which the smaller infinite class is merely a special case, for example, generalizing from the theorem of Pythagoras for right triangles to the cosine rule, which applies to any triangle. Or using the geometric meaning of the theorem of Pythagoras, since the squares on the sides of the right triangle are similar to each other, we can generalize to a relationship involving the areas of any similar figures on the sides, for example, half-circles, or regular polygons, etc.

Investigation 1

How are mathematical generalizations discovered or invented? Let us consider the familiar high school result that the opposite angles of a (convex) cyclic quadrilateral are supplementary. An obvious question to ask is what happens when we have a cyclic hexagon (compare De Villiers, 1999)? Are the opposite angles also supplementary? Or is there another relationship? This is the sort of investigation that would immediately interest the mathematically talented.

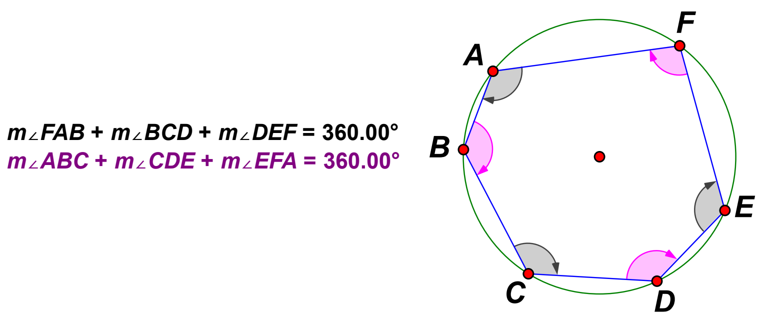

Figure 1: Cyclic hexagon

Quickly making a construction in dynamic geometry, or through deductive reasoning, students can discover the conjecture for themselves as shown in Figure 1 that the two sums of the *alternate* angles of a (convex) cyclic hexagon are both equal to 360°. More over, it shows learners that mathematics is not static and fixed as this automatically leads to an alternative rewording for the result for a cyclic quadrilateral as follows: the *alternate* angles of a (convex) cyclic quadrilateral are supplementary. The proof for the cyclic hexagon is easy, and talented students will almost immediately find it by decomposing the cyclic hexagon into two cyclic quadrilaterals by drawing say, diagonal *AD*. Obviously the next question is to investigate what happens to a cyclic octagon, from which students can then make the further generalization to any (convex) cyclic $2n$-gon for which the sum of alternate angles are equal to $(n - 1)180°$.

It is perhaps interesting to note that the final generalization generalizes to an infinite class, which contains an infinite number of infinite classes.

But what about the converse for the above generalization to any convex cyclic $2n$-gon? Is it generally true that if a sum of alternate angles of a $2n$-gon is equal to $(n -1)180^\circ$, then it is cyclic?

During my teaching experience with mostly prospective or in-service high school teachers, where I usually deal with this question after we've proved the converse result for a quadrilateral, it is seldom that they exhibit any doubts about it being generally true. Typical comments are something like: "If it is true for quadrilaterals, which we have just shown, then it obviously **has** to be true also for all $2n$-gons!"

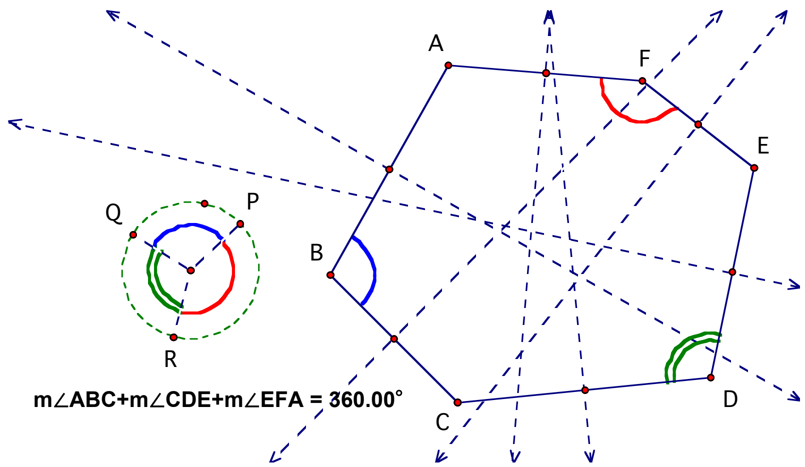

Figure 2: Refuting the converse case for a hexagon

Indeed there is often a tangible sense of annoyance, exasperation or boredom in the class when asked to check whether it is indeed true for, say, a hexagon. Nevertheless, encouraging them to check experimentally, eventually leads to some of them making the startling discovery (to them) that it is not necessarily true for a hexagon. Starting with three angles adding up to 360˚ by arranging them around a point as shown in Figure 2, it is easy to construct a dynamic hexagon with three alternate angles correspondingly equal to these angles. For example, consider the hexagon *ABCDEF* shown in the figure where $\angle B + \angle D + \angle F = 360°$, but the hexagon is not cyclic since the perpendicular bisectors of its sides are not concurrent! So there is no equidistant point in relation to all six vertices, and therefore no circle can be drawn through all six vertices!

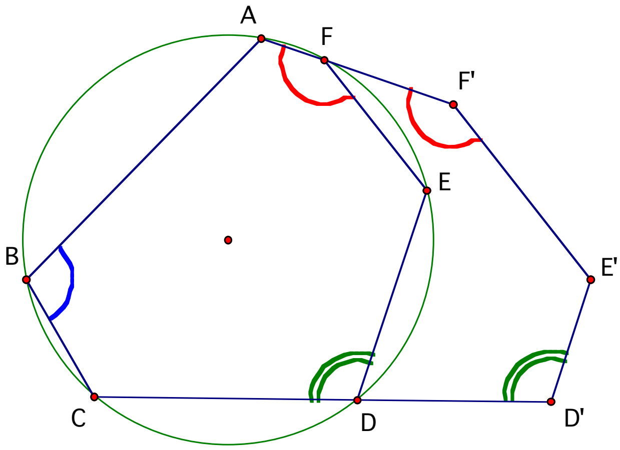

Figure 3: An elegant refutation of the converse case for a hexagon

Of interest here is also an argument given by Werner Olivier, the top student in my 2005 geometry class for prospective high school teachers. Instead of constructing a dynamic hexagon as described in Figure 2, he considered any cyclic hexagon as shown in Figure 3. Then he respectively extended *AF* and *CD* to *F'* and *D'*, and then drew lines through these points respectively parallel to *FE* and *DE* to intersect in *E'*. Clearly, all the angles of *ABCD'E'F'* are the same as that of *ABCDE*, but it is obviously not cyclic, and therefore a counter-example. Now that's the sign of mature mathematical reasoning!

One of the problems with the traditional Euclidean approach to geometry is that there are very few cases where the converses of theorems are false, and students inevitably assume, or develop the misconception, that geometric converses are always true. It is therefore a valuable strategy to go beyond dealing only with the special cases of triangles and quadrilaterals as in Euclid's Elements, and to regularly examine analogous cases for polygons where appropriate. This often provides ample opportunities for showing the difference between a statement and its converse, and often highlights the "specialness" of triangles and quadrilaterals. Moreover, genuine mathematical research involves both proving and disproving, and both these need to be reflected in our teaching, especially for mathematically talented learners. It is simply not sufficient to just focus on developing students' skills in proving true statements, but not provide instructive opportunities for also developing their ability to find counter-examples.

Investigation 2

This brings us to the next investigation: How can we generalize a square to a pentagon, hexagon, septagon, octagon, etc.? What would be equivalent to the concept of 'square'?

A natural place to start is with the (usual) definition of a square[1], namely: "A square is a quadrilateral with equal sides and equal angles." Generalizing this definition obviously leads to the well-known concept of a 'regular polygon' as a polygon with all sides and angles equal. Students

[1] The mathematical process of creating a new definition (of a new concept) by starting from a known definition, and then altering, changing, generalizing, etc. that definition is known as *constructive* ('a priori') defining (e.g. see Krygowska, 1971; Freudenthal, 1973; Human, 1978; De Villiers, 2009).

could then explore the common properties of regular polygons, finding for example, that they all have equal (main) diagonals, are both cyclic and circumscribed, have line and rotational symmetry, etc.

But the next one is likely to be not so familiar to students: How can we generalize the concept of a rectangle to a hexagon, octagon, etc.? What would be the hexagonal equivalent to the concept of 'rectangle'? Once again it is a probably a good place to start once more with the (usual) definition of a rectangle, namely, "A rectangle is a quadrilateral with equal angles." Again using the process of *constructive* defining, students are likely, as I've experienced in class, to suggest a polygon with all angles equal, say an *equi-angled* polygon.

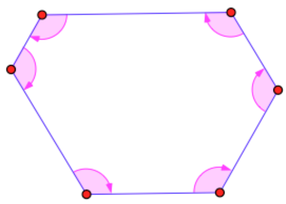

Figure 4: Equi-angled hexagon

But is this really a good generalization of the concept of rectangle? Consider the equi-angled hexagon shown in Figure 4 where all angles are equal to 120º. While the figure still has opposite sides parallel (and one can ask students to prove that!), it clearly does not have the following properties of a rectangle, namely, equal diagonals, nor is it cyclic, has no sides equal, no lines of symmetry or rotational symmetry[2].

So this appears after all to not be a good generalization of the concept of a rectangle to a hexagon. **Generally, we would like to retain as many of the properties of a rectangle as possible in its generalization.** Maybe we could add another property of a rectangle to the definition so it retains more of the rectangle's properties? But which one?

After giving learners a bit of time for reflections, they might come up with various suggestions. One suggestion I recently had in a Math Club with some high school learners (Sept 2016) following the approach outlined here was the suggestion by one learner to add the condition (like in a rectangle) that apart from equal angles, we should have opposite sides equal as well. To check this, we explored his suggestion by jointly constructing a pathological example of such a hexagon as shown in Figure 5. Though this hexagon with equal opposite sides, as well as all angles equal appears visually more 'regular' like a rectangle, has rotational symmetry of order 2 just like the rectangle, and hence bisecting diagonals, it is not cyclic, and still does not necessarily have lines of symmetry like a rectangle, nor are its diagonals necessarily equal.

[2] However, equi-angular polygons have many other interesting properties that could also be explored by mathematically talented learners. Several of these properties are discussed and presented in Ball (2002).

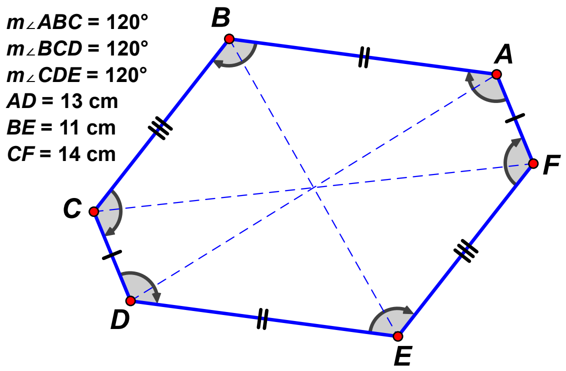

Figure 5: Equi-angled parallelo-hexagon

So though it is an improvement on the 1st attempt, it does not yet have all the critical properties of a rectangle. At this point I suggested to the learners: Let's suppose we add the condition that apart from being equi-angled, it has to be cyclic also.

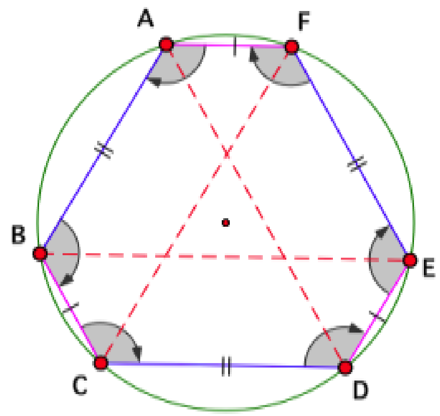

Figure 5: Cyclic equi-angled hexagon

A good idea might be to again let students construct a hexagon as shown in Figure 6, either by hand and paper and pencil or by using dynamic geometry software. Much to their surprise and delight, the learners in the Math Club I was referring to now noticed that like a rectangle the constructed cyclic, equi-angled hexagon has the following corresponding properties, namely: *alternate* (not opposite) sides equal; main diagonals equal; 3 lines of symmetry, rotational symmetry of order 3 (not 2), etc. This now looks like a good generalization[3] of the concept of a rectangle!

This now provides a rich context for posing several modestly challenging conjectures to students to prove such as: "If a cyclic hexagon has all angles equal, then the two sets of alternate sides are equal". This result can easily be proven in various ways by students and is a good example of a simple task that can lead to multiple solutions that is educationally valuable because each proof

[3] Notice that with this generalization we have, however, lost the property of diagonals bisecting each other as in Figure 5. As pointed out in Foster & De Villiers (2015, p. 9) it is important to note that it is not always possible to retain all the properties of a particular mathematical object or process when we generalize. In fact, generally, since the process of generalization mostly requires the relaxation of conditions, there are fewer special properties in a general case.

involves the application of different concepts. For example, compare the following two different proofs.

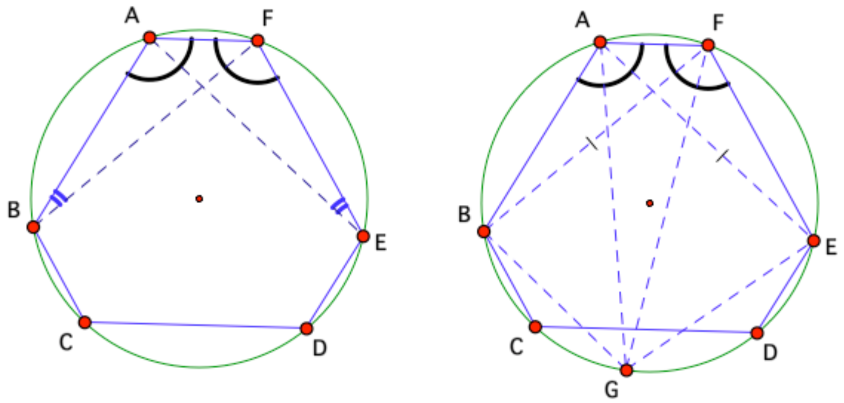

Figure 7: Two different proofs

First proof

With reference to the first diagram in Figure 6, it is easy to immediately see that ΔABF is congruent to ΔFEA ($\angle$, $\angle$, s), since $\angle ABF = \angle FEA$ on chord AF, $\angle BAF = \angle EFA$ is given, and AF is common. Hence, (alternate sides) $AB = FE$. Applying the same argument at other adjacent vertices, it follows that the two sets of alternate angles are equal, e.g. $AB = CD = FE$ and $BC = DE = FA$.

Second proof

With reference to the second diagram in Figure 6, construct an arbitrary point G on arc CD. It now follows from the given $\angle A = \angle F$ that chord BF = chord EA (equal angles subtend equal chords). Using the same theorem, it follows that $\angle BGF = \angle EGA$, and subtracting the common $\angle AGF$ from both sides of this equality, gives $\angle BGA = \angle EGF$. Hence, the chords subtended by these two angles are equal, e.g. $AB = FE$. As above, the same argument applies to the other adjacent vertices.

Though the second proof is slightly longer, it avoids having to use congruency. This result can be proved in other ways as well, and is therefore a good example of a Multiple Solution Task, which Leikin (2011), Levav-Waynberg & Leikin (2012), and others have found productive in promoting creative thinking among students at university and high school level.

The above result for a cyclic hexagon with equal angles naturally leads to the general theorem as discussed in De Villiers (2011a) that “If a cyclic $2n$-gon has all angles equal, then the two sets of alternate sides are equal”, and where cyclic $2n$-gons with all angles equal have been called *semi-regular angle*-gons.

One could also fruitfully challenge students to consider the converse result, namely, if the two sets of alternate sides of a cyclic hexagon are equal does it imply that all its angles are equal? As reported in Samson (2015), in relation to an examination task similar to this, Grade 12 students at a South African high school successfully produced five different proofs for this converse result.

Students can also next be asked to prove that a cyclic hexagon with equal angles, like the rectangle, has equal diagonals (both major and minor). Another interesting property is that its main diagonals intersect at 60° to each other.

One could also engage learners with investigating different possible definitions for these ‘regular angle-gons’ and critically comparing them in terms of ease of use and/or deductive convenience (how easy or difficult is it to derive the other properties not contained in the definition). Apart from defining these *semi-regular angle-gons* as cyclic $2n$-gons with all angles equal, they could for example be defined as cyclic $2n$-gons with alternate sides equal, a cyclic $2n$-gon with rotational symmetry of order n, or as $2n$-gons with n axes of symmetry through n pairs of opposite sides, etc. In this way, rather than just providing learners with ready-made definitions as in the traditional approach, learners can develop a better understanding of where definitions come from, and the criteria used for eventually selecting one definition over another.

CONCLUDING REMARKS

Following up on the preceding activity, learners could next be asked to similarly explore the ‘constructive’ generalization of the concept of a rhombus to higher polygons, or that of an isosceles trapezium, kite (e.g. see De Villiers, 2011b), or that of a parallelogram (e.g. see De Villiers, 2009), etc.

However, there are many other suitable concepts and theorems in secondary geometry, such as Varignon’s theorem (see De Villiers, 2007), for example, that could equally well be used as stimulating starting points to engage students in the exciting process of further generalization beyond the narrow confines of the prescribed curriculum.

While generalization is a powerful tool for the discovery of new mathematics, and establishing new links between previously disconnected concepts, theorems or topics, students are also likely to come up with some false generalizations of their own (compare Foster, 2013), but this creates a learning opportunity for further exploration through experimentation and producing counter-examples.

Looking back or further reflecting on the solution of a problem (often a general proof) is the last step in Polya’s problem-solving heuristic (Polya, 1945), but has largely been neglected as a research area in mathematics education. In classrooms, for example, the majority of students and teachers usually just move on to the next problem, without spending some time critically analyzing a solution or proof. This is a pity as it is often possible to immediately further generalize a result by identifying the essential characteristic on which it is based. For example, Leong et al (2012) describe some success using a worksheet based on Polya’s model to guide a high achieving student to ‘look back’ at his solution and push him to further extend, adapt and generalize his solution. This reflective process leading to a subsequent generalization illustrates what has been broadly termed the ‘discovery’ function of proof (De Villiers, 1990) as well as a ‘deductive’ generalization (De Villiers, 2012; Govender, 2013), and is also a rich, potential area for possible further research with mathematically talented learners.

As argued by Polya (1954, p. 30), while quoting Schopenhauer, we understand a mathematical relationship or concept more broadly and more purely when we recognize it as the same in widely different cases and between completely heterogeneous objects. As already mentioned at the start, it is therefore a potentially valuable experience for our mathematically talented students to engage them in activities like these as they might go on to become active and productive research mathematicians one day.

References

Ball, D. (2002). Equiangular polygons. *The Mathematical Gazette*, Vol. 86, No. 507 (Nov.), pp. 396-407.

Bailey, L.M., Morris, L.G., Thompson, W.D., Feldman, S.B., & Demetrikopoulos, M.K. (2016). Historical contribution of creativity to development of gifted science education in formal and informal learning environments. In M.K. Demetrikopoulos & J.L. Pecore (Eds.), *Interplay of Creativity and Giftedness in Science*, pp. 3-13. Rotterdam: Sense Publishers.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, pp. 7−24. Accessed January 2016 from: http://mysite.mweb.co.za/residents/profmd/proofa.pdf

De Villiers, M. (1999). *Rethinking Proof with Sketchpad*. Emeryville: Key Curriculum Press.

De Villiers, M. (2007). A hexagon result and its generalization via proof, *The Montana Mathematics Enthusiast*, June 2007, Vol. 4, No. 2, pp. 188-192.

De Villiers, M. (2009). *Some Adventures in Euclidean Geometry*. Raleigh, NC: LULU Publishers.

De Villiers, M. (2011a). Equi-angled cyclic and equilateral circumscribed polygons. *The Mathematical Gazette*, 95(532), March, pp. 102-106. Accessed 16 October 2016 at: http://dynamicmathematiclearning.com/equi-anglecyclicpoly.pdf

De Villiers, M. (2011b). Feedback: Note 95.14, Equi-angled cyclic and equilateral circumscribed polygons. *The Mathematical Gazette*, 95(533), July 2011, p. 361. Accessed 16 October 2016 at: http://dynamicmathematicslearning.com/equi-angle-equi-side-general.pdf

De Villiers, M. (2012). An illustration of the explanatory and discovery functions of proof. *Pythagoras*, 33(3), Art. #193, 8 pages. http://dx.doi. org/10.4102/pythagoras. v33i3.193

Foster, C. (2013). Problems with generalising: Pythagoras in N dimensions. *Australian Senior Mathematics Journal*, Vol. 27, No. 1, pp. 8-11.

Foster, C. & De Villiers, M. (2015). The definition of the scalar product: an analysis and critique of a classroom episode. *International Journal of Mathematical Education in Science and Technology*, http://dx.doi.org/10.1080/0020739X.2015.1117148

Freudenthal, H. (1973). *Mathematics as an Educational Task*. D. Reidel, Dordrecht, Holland.

Govender, R. (2013). Constructions and Justifications of a Generalization of Viviani's Theorem. Unpublished PhD dissertation, University of KwaZulu-Natal. Accessed January 2016 at: http://dynamicmathematicslearning.com/Constructions-Justifications-Viviani-generalization-thesis.pdf

Human, P. G. (1978). Wiskundige werkwyses in wiskunde-onderwys. (Mathematical processes in mathematic education). Unpublished doctoral dissertation, Stellenbosch University, South Africa.

Johnston-Wilder, S. & Mason, J. (2005). *Developing Thinking in Geometry*. London: Paul Chapman Publishing.

Krygowska, A.Z. (1971). Treatment of the Axiomatic Method in Class. In Servais, W. & Varga, T., *Teaching School Mathematics*, Penguin-Unesco, London, 124-150.

Levav-Waynberg, A. & Leikin, R. (2012). The role of multiple solution tasks in developing knowledge and creativity in geometry. *Journal of Mathematical Behavior*, 31, pp. 73–90. doi:10.1016/j.jmathb.2011.11.001

Leikin, R. (2011). Multiple-solution tasks: from a teacher education course to teacher practice. *ZDM Mathematics Education*, 43, pp. 993–1006. DOI 10.1007/s11858-011-0342-5

Leikin, R. & Pitta-Pantazi, D. (2013). Creativity and mathematics education: the state of the art. *ZDM Mathematics Education*, 45, pp. 159–166. DOI 10.1007/s11858-012-0459-1

Leong, Y.H., Toh, T.L., Tay, E.G., Quek, K.S., & Dindyal, J. (2012). Relooking 'look back': A student's attempt at problem solving using Polya's model. *International Journal of Mathematical Education in Science and Technology*, 43(3), 357−369. http:// dx.doi.org/10.1080/0020739X.2011.618558

Lockhart, P. (2002) A mathematician's lament. In Devlin's Angle (March 2008) at the Mathematical Association of America website. Available from: http://www.maa.org/devlin/LockhartsLament.pdf

Mann, E.L. (2006). Creativity: The Essence of Mathematics. *Journal for the Education of the Gifted*, 30(2), pp. 236-260.

Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Polya, G. (1954). *Mathematics and Plausible Reasoning*, Vol. 1. Princeton, NJ: Princeton University Press.

Samson, D. (2015). Devising Explorative Euclidean Geometry Questions. *Learning and Teaching Mathematics*, no. 19, pp. 13-16.

Sheffield, L. J. (ed.) (1999). *Developing Mathematically Promising Students*. Reston, VA: National Council of Teachers of Mathematics.

Sriraman, B. (2004). The Characteristics of Mathematical Creativity. *The Mathematics Educator*, Vol. 14, No. 1, 19–34.

Stewart, I. (2008). *Taming the Infinite*. London: Quercus Publishing.