Marking scheme for P4 (G1 in shortlist)

General guidelines

Because of the variety of approaches to problem 4, we (the Coordinating Team for this problem) have written marking guidelines for a number of approaches. In order to unify these, and also to cover any solutions that we have not thought of, we offer this general statement of philosophy to be used in grading this problem.

1. We look for a first step that is likely to lead to a solution, and award one or two points for it. If this step is not one that we have already recognized we look to the leader to offer an explanation of how this step will lead to a solution.
2. The remainder of the marks will be given for a detailed argument that leads to a solution. Partial marks will be considered in the development of a student's argument.
3. In certain cases, final details may be deemed significant, and papers missing these details will be denied at most one mark.

Additivity

- Within an approach the points given for the major steps are additive.
- In case of presence of different incomplete solutions the maximum is taken.

Standalone facts that are worth 0 point

- Stating that the problem is equivalent to proving that $\angle BSC + \angle BAC = 180°$
- Considering particular cases of triangle ABC

Official solution 1

1. Stating that triangles ABP and CAQ are similar .... 1 point
2. **Proving** that triangles BPM and NQC are similar .... 2 points
3. **Proving** that BPM, NQC and BSC are all similar OR $CPSM$ is cyclic .... 2 points
4. Conclusion .... 2 points

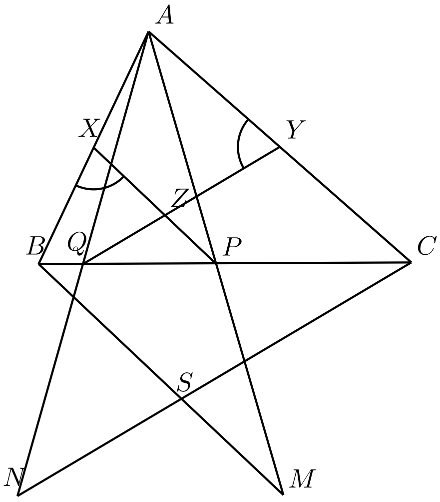
Official solution 2

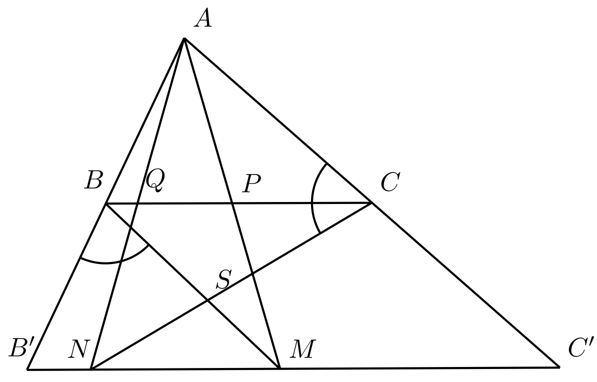
1. Proving that $\angle LBC = \angle CBA$ OR $\angle KCB = \angle BCA$ .... 1 point
2. Stating and proving that X lies on line MN .... 3 points
3. Applying Pascal's theorem (the hexagon must be explicitly written, in the right order) .... 3 points

One can also define X as the reflection across BC and prove that K and L (the intersection points of CX and AM and BX and AN, respectively) are on the circumcircle. In this case, the marking scheme is

1. Stating and proving that X lies on line MN .... 1 point
2. Proving that K and L lies on the circumcircle .... 3 points
3. Applying Pascal's theorem (the hexagon must be explicitly written, in the right order) .... 3 points

Similarity and medians (by ROM7, MEX7, EST7)



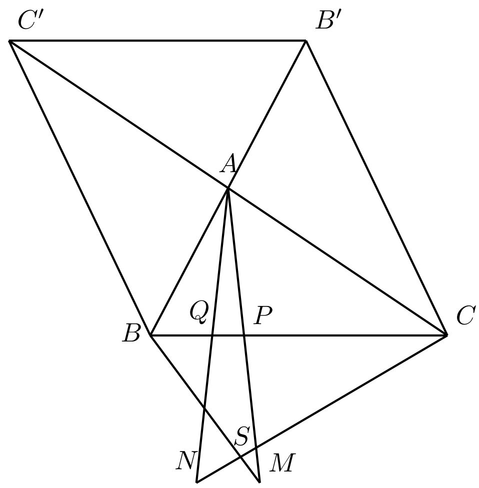


1. Stating that triangles ABP and CAQ are similar . . . . 1 point
2. Proving that $AXZY$ is cyclic . . . . 4 points
3. Using homothety . . . . 2 points

In the case of the solutions by MEX7 and EST7 (see diagram on the right), the breakdown would be

1. Constructing points B' and C' AND stating that M, N, B', C' are collinear . . . . 2 points
2. Stating that triangles $AB'M$ and $C'AN$ are similar . . . . 1 point
3. Proving that $ABSC$ is cyclic . . . . 4 points

Parallelogram (by ISR7 and SAU Observer)



1. Stating that triangles APB and CAB are similar . . . . 1 point
2. Proving that BAM and BCC' are similar . . . . 4 points
3. Proving that $\angle CBS + \angle SCB = \angle BAC$ . . . . 2 points

Complex numbers

As usual, we represent the corresponding complex number by the corresponding lowercase letter. Proofs for the following facts are not required:

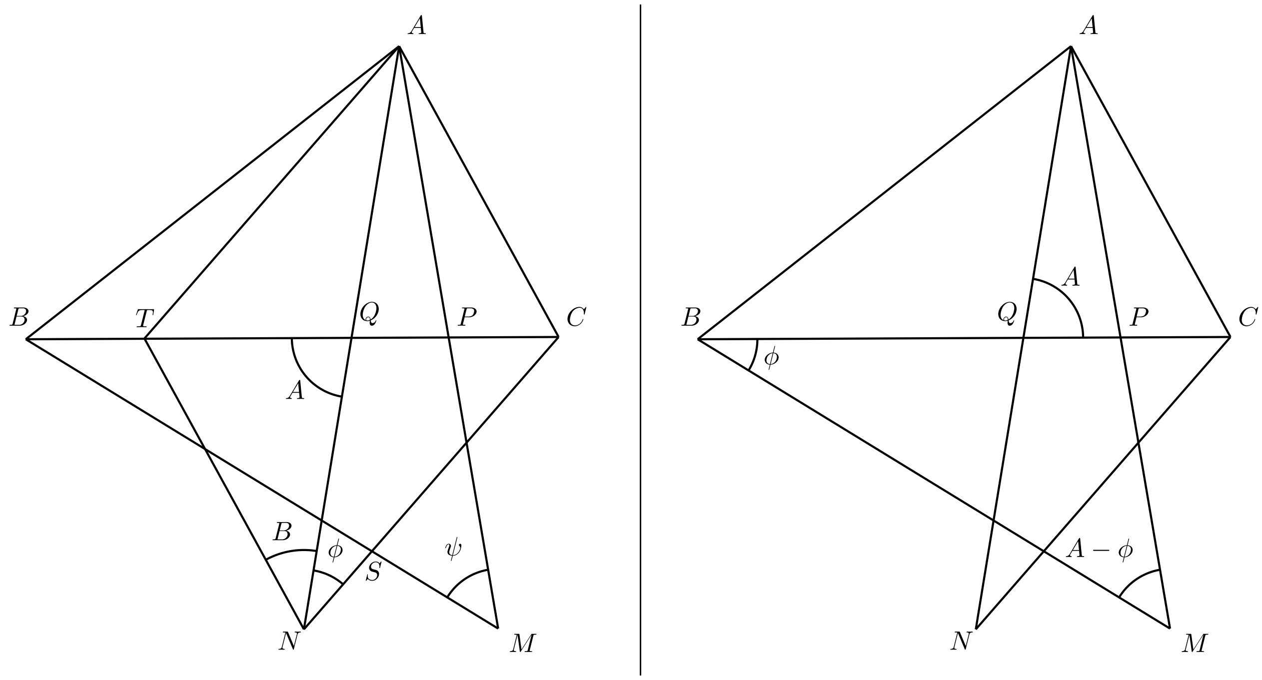
- The orthocenter of ABC is $h = a + b + c$;
- Equations of a general line or a general circle;
- The equation of a chord AB in the unit circle is $z + ab\overline{z} = a + b$;
- The formula for the projection of a point P onto a chord AB: $\frac{1}{2}(a + b + p - ab\overline{p})$.

Here are some checkpoints for this problem:

- $p = a + \frac{(b-a)(c-a)b}{a(c-b)}$;
- $m = a + \frac{2(b-a)(c-a)b}{a(c-b)}$;
- Let $S_1 \neq B$ be the intersection of line BM and the circumcircle of the triangle ABC. So $s_1 = \frac{ab+ac-2bc}{2a-b-c}$. Of course, s (S being the intersection of BM and CN) is the same.

1. Finding m in terms of a, b and c .... 2 points
2. Conclusion .... 5 points

Trigonometry approaches



1. Finding $\cot\phi$ and $\cot\psi$ in terms of $\angle A$, $\angle B$ and $\angle C$ or equivalent .... 2 points
2. Proving that $\cot(\phi + \psi) = \cot A$ or $\cot\phi = \cot(A - \psi)$ or any equivalent equation[1] .... 4 points
3. Concluding that $\angle A = \phi + \psi$, by considering the appropriate intervals .... 1 point

[1] Finding the ratio $\frac{\sin(A-\phi)}{\sin\phi} = \sin A \cot\phi - \cos A$ is equivalent to this.

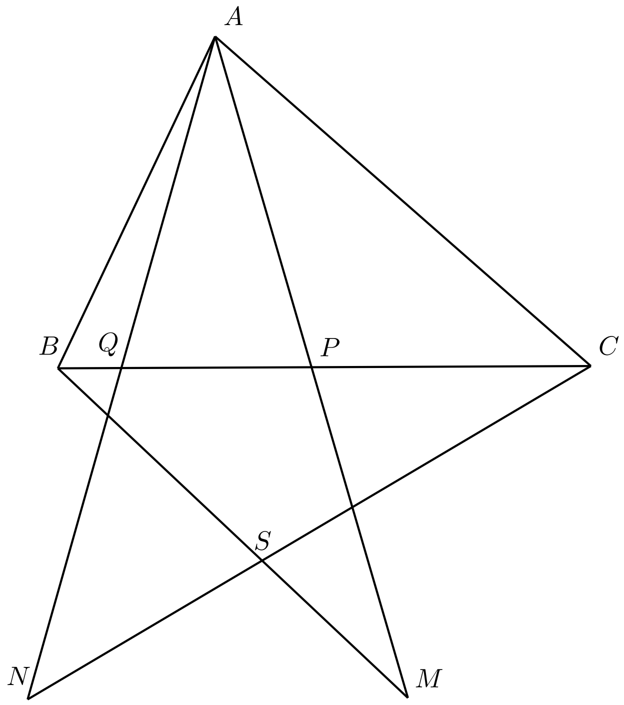
Areal methods (by UNK7)

Again, here are some checkpoints:

- The minus power function of Γ_B is $\sigma^2 - c^2 y$;
- $P = (0; 1 - \frac{c^2}{a^2}; \frac{c^2}{a^2})$. and $Q = (0; \frac{b^2}{a^2}; 1 - \frac{b^2}{a^2})$;
- $M = (-1; 2(1 - \frac{c^2}{a^2}); 2\frac{c^2}{a^2})$ and $N = (-1; 2\frac{b^2}{a^2}; 2(1 - \frac{b^2}{a^2}))$;
- The line BM has equation $2\frac{c^2}{a^2}x + z = 0$ and the line CN has equation $2\frac{b^2}{a^2}x + y = 0$. S has un-normalized areal coordinates $(-a^2 : 2b^2 : 2c^2)$.

1. Finding P and Q in terms of a, b and c .... 2 points
2. Conclusion .... 5 points

Power of a point



1. Reducing the problem to proving that $BS \cdot BM = AB^2$ .... 2 points
2. Conclusion .... 5 points

Coordinates

If $A = (0, a)$, $B = (b, 0)$ and $C = (c, 0)$, then

- $a \cot A = \frac{a^2 + bc}{c - b}$;
- $P = (a \cot A, 0)$ and $Q = (-a \cot A, 0)$;
- $M = (2a \cot A, -a)$ and $N = (-2a \cot A, -a)$;

- $S=\left(\frac{2(b+c)(a^2+bc)}{4a^2+(b+c)^2};-\frac{a(b-c)^2}{4a^2+(b+c)^2}\right)$;
- The circumcircle's equation is $a(x^2+y^2-(b+c)x+bc)-(a^2+bc)y=0$.

1. Finding the coordinates of S in terms of A, B and C .... 2 points
2. Conclusion .... 5 points