An Extension of an IMO 2014 Geometry Problem

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INTRODUCTION

It was a great personal privilege and experience to attend and participate as a coordinator at the 55th International Mathematical Olympiad (IMO), which was held from 3-13 July 2014 in Cape Town. It was also historic because it was the first time the IMO was held on the African continent. Here is the first problem written on the second day of the competition (problem no. 4):

> "The points P and Q are chosen on the side BC of a triangle ABC so that $\angle PAB = \angle ACB$ and $\angle QAC = \angle CBA$. The points M and N are taken on the rays AP and AQ, respectively, so that $AP = PM$ and $AQ = QN$. Prove that the lines BM and CN intersect on the circumcircle ABC."

Note that unlike our South African matric examination papers, no diagrams are given in the IMO – learners are expected to draw their own. In addition, the diagrams do not count for marks; only the proofs do.

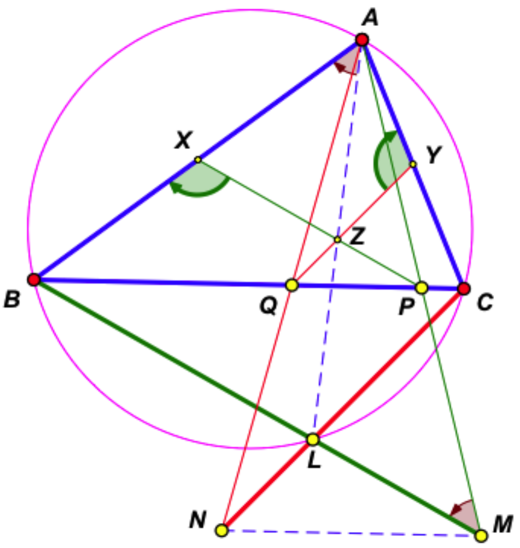

FIGURE 1: IMO 2014 problem no. 4.

Suppose the point of intersection of the lines BM and CN is L (see Figure 1). One of the IMO students from Macau observed the interesting property, en route to eventually proving the required result, that the products of the opposite (alternate) sides of the cyclic quadrilateral $ABLC$ were equal, and hence according to Ptolemy's theorem[1], equal to half the product of the diagonals, i.e. $AB \times CL = BL \times AC = \frac{1}{2} AL \times BC$. A dynamic geometry sketch illustrating this latter result and the original IMO problem is available at: http://www.dynamicmathematicslearning.com/IMO-2014-prob4.html

[1] Ptolemy's theorem (approx. 100 AD) states that for a cyclic quadrilateral $ABCD$ the following holds true: $AD \times BC + AB \times CD = AC \times BD$.
See Man (2019) for more information or *Wikipedia* at: https://en.wikipedia.org/wiki/Ptolemy%27s_theorem

The purpose of this paper is to discuss an interesting further extension of this latter property to hexagons, octagons, etc. by using the same construction method to construct points similar to point L on the other sides of triangle ABC. However, let us first present a proof of the original IMO problem as well as of the result about the sides and diagonals of the constructed cyclic quadrilateral $ABLC$.

PROOF OF IMO PROBLEM

The beauty of this problem is that it can be proved in many different ways ranging from synthetic approaches involving similarity, Pascal's theorem, enlargement, power of a point, parallelograms, etc. to computational approaches involving areal or Cartesian coordinates, trigonometry, etc. The proof below using a homothetic (similarity) transformation, and given by some IMO students, is particularly elegant.

Let X and Y be the midpoints of sides AB and AC respectively, and Z the intersection of PX and QY as shown in Figure 1. Then PX and QY are medians of similar triangles ABP and CAQ. Thus, $\angle BXP = \angle AYQ$ and therefore quadrilateral $AXZY$ is cyclic (exterior angle equals opposite interior angle). Now the homothety (enlargement) with centre A and ratio 2 maps $AXZY$ to $ABLC$, and completes the proof.

PROOF OF SIDES AND DIAGONALS RELATIONSHIP OF CYCLIC $ABLC$

From the similar triangles PBA and ABC in Figure 1, we get that $PA = \frac{BA \times AC}{BC}$; hence $MA = \frac{2BA \times AC}{BC}$ …(1). Let $\angle LAB = x$. From the triangle sum in triangle ABL, and that $\angle ALB = \angle ACB$ on the circumcircle, it follows that $\angle CBL = \angle BAC - x$. But $\angle BPA = \angle BAC$. Therefore, $\angle AMB = \angle BPA - \angle CBL = x$. This implies that triangles ABL and MBA are also similar (equiangular). Hence $\frac{BL}{AL} = \frac{BA}{MA}$. Thus, with substitution of (1) we obtain:

$$AC \times BL = AC \times \left(\frac{BA \times AL}{MA}\right) = AC \times \left(\frac{BA \times AL}{1} \times \frac{BC}{2BA \times AC}\right) = \frac{1}{2} AL \times BC$$

From Ptolemy's theorem it therefore follows that $AB \times CL = BL \times AC = \frac{1}{2} AL \times BC$.

EXTENSION TO CYCLIC HEXAGONS

What happens if points N and M are constructed on sides AB and AC, respectively, in the same way as L as shown in Figure 2?

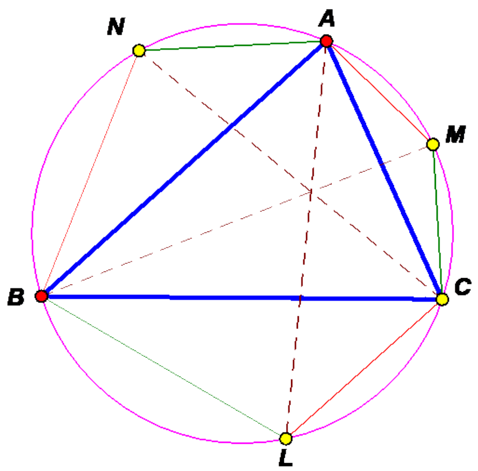

FIGURE 2: Extending the idea to cyclic hexagons.

As quickly confirmed by a dynamic geometry sketch we analogously obtain that the two products of the *alternate* sides of the cyclic hexagon $ANBLCM$ are equal, as well as one eighth of the product of the main diagonals. In other words, $NB \times LC \times MA = BL \times CM \times AN = \frac{1}{8} AL \times BM \times CN$ as shown with the online dynamic sketch at: http://www.dynamicmathematicslearning.com/IMO2014-extension.html

PROOF OF SIDES AND DIAGONALS RELATIONSHIP OF CYCLIC $ANBLCM$

Using the sides and diagonals result for $ABLC$, we now have the following three relationships in Figure 2:

- $AB \times LC = BL \times AC = \frac{1}{2} BC \times AL$ … (for cyclic $ABLC$)
- $BC \times MA = AB \times CM = \frac{1}{2} AC \times BM$ … (for cyclic $ABCM$)
- $AC \times NB = BC \times AN = \frac{1}{2} AB \times CN$ … (for cyclic $NBCA$)

Using the first and third expressions of each of these relations we have:

$$AL \times BM \times CN = \frac{2AB \times LC}{BC} \times \frac{2BC \times MA}{AC} \times \frac{2AC \times NB}{AB} = 8(NB \times LC \times MA)$$

Alternately, using the second and third expressions of each of these relations we have:

$$AL \times BM \times CN = \frac{2BL \times AC}{BC} \times \frac{2AB \times CM}{AC} \times \frac{2BC \times AN}{AB} = 8(BL \times CM \times AN)$$

This proves the equality relationship between the products of the alternate sides and their relationship with the product of the main diagonals.

Another interesting property of the constructed hexagon $ANBLCM$ as shown in Figure 2 is that the main diagonals AL, BM and CN are concurrent. This follows immediately from the following lovely theorem by Cartensen (2000-2001), a proof of which is provided at the end of the article as an appendix, which states that the main diagonals of a cyclic hexagon are concurrent if and only if the two products of alternate sides are equal, i.e. $NB \times LC \times MA = BL \times CM \times AN$. This theorem also appears in Gardiner & Bradley (2005, p. 96; 99) and also on the *Math Stack Exchange* (2013).

FURTHER EXTENSION TO CYCLIC $2n$-GONS

If we similarly construct points O and P respectively in relation to triangles BAN and BCL and on opposite arcs AN and LC as shown in Figure 3, we obtain the same relationship between the alternate sides and the main diagonals of the formed octagon. However, in this case it is:

$$OP \times NC \times BM \times LA = 64(AO \times NB \times LP \times CM) = 64(ON \times BL \times PC \times MA)$$

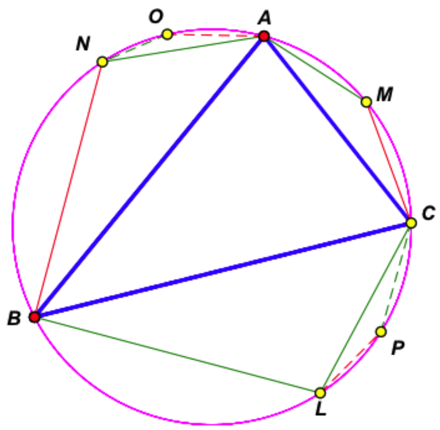

FIGURE 3: Extending the idea to cyclic $2n$-gons.

By continuing to add pairs of points on opposite arcs in the same way, the result can be extended to $2n$-gons in general. For example, if as illustrated in the dynamic sketch at the preceding URL, we similarly construct points Q and R respectively on opposite arcs AO and PL of the octagon in Figure 3, we obtain the following for the formed decagon:

$$QR \times OP \times NC \times BM \times LA = 1152(AQ \times ON \times BL \times RP \times CM)$$
$$= 1152(QO \times NB \times LR \times PC \times MA)$$

Proofs of these last two results, and further exploration of these, as well as of other variations, are left to the reader.

REFERENCES

Cartensen, J. (2000-2001). About hexagons. *Mathematical Spectrum*, 33(2), pp. 37–40.

Gardiner, A.D. & Bradley, C.J. (2005). *Plane Euclidean Geometry: Theory and Problems*. Leeds, University of Leeds: The United Kingdom Mathematics Trust.

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APPENDIX – PROOF OF CARTENSEN'S THEOREM

Suppose that the main diagonals of a cyclic hexagon intersect at O as shown in Figure 4a. Then triangles ABO and EDO are similar, with $\frac{AB}{DE} = \frac{AO}{EO}$. Similarly, the other two pairs of similar triangles give $\frac{CD}{FA} = \frac{CO}{AO}$ and $\frac{EF}{BC} = \frac{EO}{CO}$. Multiplied together, these give the desired product.

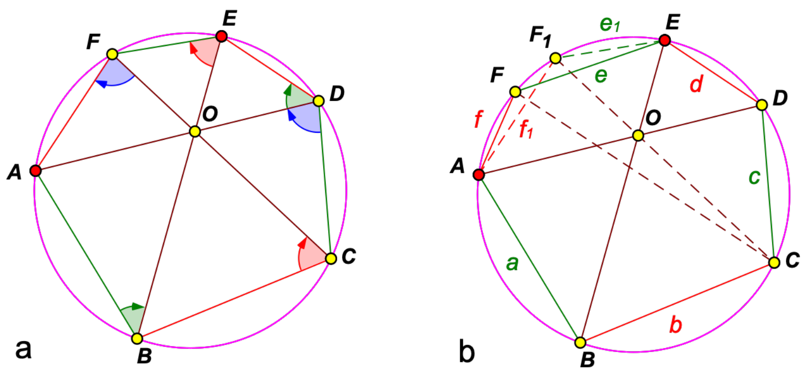

FIGURE 4: Proof of Cartensen's theorem.

Conversely, suppose the product holds, i.e. $ace = bdf$ as shown in Figure 4b. Let BE and AD intersect at O. Extend CO to intersect the circle at F_1. From the preceding proof, we now have $ace_1 = bdf_1$. The two equalities so far provide $acef_1 = bdff_1 = acfe_1$ which can simplify to give $ef_1 = fe_1$. From Ptolemy's theorem on cyclic quadrilaterals on $AEFF_1$ we get $ef_1 = e_1 f + FF_1 \times AE \Rightarrow FF_1 \times AE = 0$ from which we can conclude that $FF_1 = 0$, i.e. that CF also passes through O as required.