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ON SEQUENCES OF NESTED TRIANGLES

Dan Ismailescu (Hempstead) and Jacqueline Jacobs (Hempstead)

Hofstra University, Department of Mathematics,
Hempstead, NY 11549, USA

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Abstract

For a given triangle, we consider several sequences of nested triangles obtained via iterative procedures. We are interested in the limiting behavior of these sequences. We briefly mention the relevant known results and prove that the triangle determined by the feet of the angle bisectors converges in shape towards an equilateral one. This solves a problem raised by Trimble [15].

1. Introduction

For a given triangle T_0 one may construct a sequence of triangles via an iterative procedure. A simple example of this type of construction is to let T_1 be the triangle whose vertices are the midpoints of the sides of T_0, T_2 be the triangle whose vertices are the midpoints of the sides of T_1 etc. Continuing this process one obtains a nested sequence of triangles (see Figure 1).

On one hand, it is not difficult to show that, as the successive triangles become smaller, the sequence of triangles defined above converges to a *limiting point*, which in this case is the centroid of T_0.

On the other hand, as the size of the triangles gets smaller we will be interested in studying the change in their *shape*, that is, we will concentrate only on the angles of these triangles. It is easy to see that the midpoint construction yields a sequence

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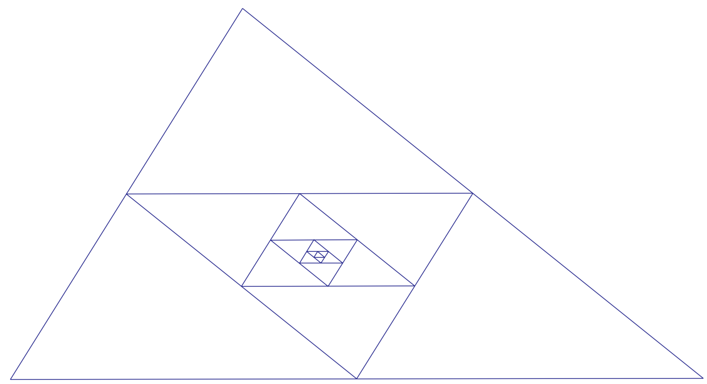

FIGURE 1. *The midpoint sequence*

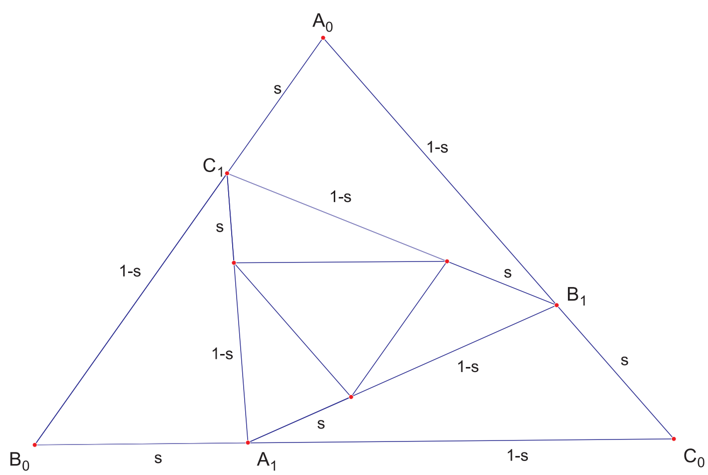

FIGURE 2. *Division ratio* $= s$

of similar triangles so the *limiting shape* is the same as the shape of the initial triangle T_0.

But what happens if instead of bisecting the sides of each triangle we consider some other subdivision ratio, $s \in (0,\, 1)$? (see Figure 2). It is relatively easy to show that the limiting point is still the centroid of T_0 but the situation can be rather complicated when it comes to the limiting shape. (a partial answer is provided in Theorem 1).

In general, given a triangle and a certain iterative process which generates a sequence of (eventually nested) triangles, there are two questions regarding the limiting behavior of this sequence we would like to answer:

QUESTION 1. Does this triangle-sequence converge to a point? If so, what are the coordinates of this limiting point?

QUESTION 2. Is this triangle-sequence convergent in shape? In other words, if $T_n = \triangle A_nB_nC_n$ is the n-th triangle of the sequence, what can be said about the sequences $\{\widehat{A}_n\}_{n\geq 0}$, $\{\widehat{B}_n\}_{n\geq 0}$,$\{\widehat{C}_n\}_{n\geq 0}$?

This kind of questions have been investigated by many authors including P. J. Davis, L. R. Hitt, M. de Villiers, J. G. Kingston, J. L. Synge, J. Ding and X. M. Zhang [2, 5, 7, 16, 12, 9, 10]. Constructing sequences of triangles and studying the geometry of their limits has repeatedly appeared in numerous journals and books [1, 3, 8, 11, 13, 14, 15].

2. Summary of known results

Below we mention some of the relevant results concerning this type of problems.

THEOREM 1 (FIXED DIVISION RATIO CONSTRUCTION). *Let s be an arbitrary fixed number between 0 and 1. Given a triangle $T_0 = \triangle A_0B_0C_0$ we consider the points A_1 on B_0C_0, B_1 on A_0C_0 and C_1 on A_0B_0 such that*

$$\frac{B_0A_1}{A_1C_0} = \frac{C_0B_1}{B_1A_0} = \frac{A_0C_1}{C_1B_0} = \frac{s}{1-s}.$$

Define a new triangle $T_1 = \triangle A_1B_1C_1$. By repeating this construction we obtain a nested sequence of triangles $\{T_n = \triangle A_nB_nC_n\}_{n\geq 0}$ (see Figure 2).

a) $\{T_n\}_{n\geq 0}$ converges to the centroid of T_0, as $n \to \infty$.

b) Let

$$\theta = \cos^{-1}\left[-1 + \frac{1}{2(1-3s+3s^2)}\right].$$

If $\theta = (p/q)\,\pi$ where p and q are positive integers, then the shape sequence is periodical. More precisely, for every $k \geq 0$ triangles $\triangle A_kB_kC_k$ and $\triangle A_{k+2q}B_{k+2q}C_{k+2q}$ are similar.

Although we were not able to locate a reference for this problem, we doubt that the result is new. We leave the straightforward proof as an exercise.

Another construction belonging to the same class is mentioned below.

THEOREM 2 ([10, 14] THE INCIRCLE-CIRCUMCIRCLE CONSTRUCTION). *Given a triangle $T_0 = \triangle A_0B_0C_0$ label by A_1, B_1, C_1 the points where the incircle of T_0 touches the sides B_0C_0, C_0A_0 and A_0B_0, respectively. Consider the new triangle $T_1 = \triangle A_1B_1C_1$* (see Figure 3). *Similarly we can form $T_2 = \triangle A_2B_2C_2$*

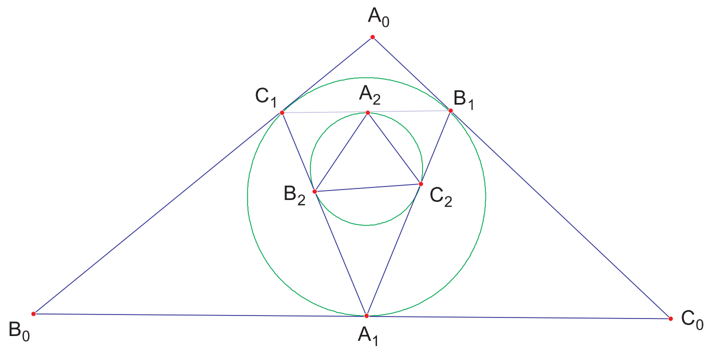

FIGURE 3. *The incircle-circumcircle sequence*

using $\triangle A_1B_1C_1$. Continuing in this fashion we construct the sequence $\{T_n = \triangle A_nB_nC_n\}_{n\geq 0}$. Then, as $n \to \infty$, T_n converges (in shape) towards an equilateral triangle.

PROOF. (Sketch)

It is easy to show that the angles of T_1 depend linearly on the angles of T_0. More precisely,

$$\begin{bmatrix}\widehat{A}_1\\ \widehat{B}_1\\ \widehat{C}_1\end{bmatrix} = \begin{bmatrix}0 & 1/2 & 1/2\\ 1/2 & 0 & 1/2\\ 1/2 & 1/2 & 0\end{bmatrix} \cdot \begin{bmatrix}\widehat{A}_0\\ \widehat{B}_0\\ \widehat{C}_0\end{bmatrix}, \quad \text{which by induction gives that}$$

$$\begin{aligned}\lim_{n\to\infty} \begin{bmatrix}\widehat{A}_n\\ \widehat{B}_n\\ \widehat{C}_n\end{bmatrix} &= \lim_{n\to\infty} \begin{bmatrix}0 & 1/2 & 1/2\\ 1/2 & 0 & 1/2\\ 1/2 & 1/2 & 0\end{bmatrix}^n \cdot \begin{bmatrix}\widehat{A}_0\\ \widehat{B}_0\\ \widehat{C}_0\end{bmatrix}\\ &= \begin{bmatrix}1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\\ 1/3 & 1/3 & 1/3\end{bmatrix} \cdot \begin{bmatrix}\widehat{A}_0\\ \widehat{B}_0\\ \widehat{C}_0\end{bmatrix} = \begin{bmatrix}\pi/3\\ \pi/3\\ \pi/3\end{bmatrix}.\end{aligned}$$

Therefore the limiting shape is an equilateral triangle. $\square$

It is interesting to note that the triangle sequence $\{T_n\}_{n\geq 0}$ defined above has also a limiting point. (the radius of each new circle is at most half the radius of the previous circle). In fact, this limit point is called the *Poncelet* point of the initial triangle T_0. Synge (see [6], problem B25) asks whether this can be specified in finite terms (i.e., by a formula involving the vertices of T_0). We do not know the answer to this question.

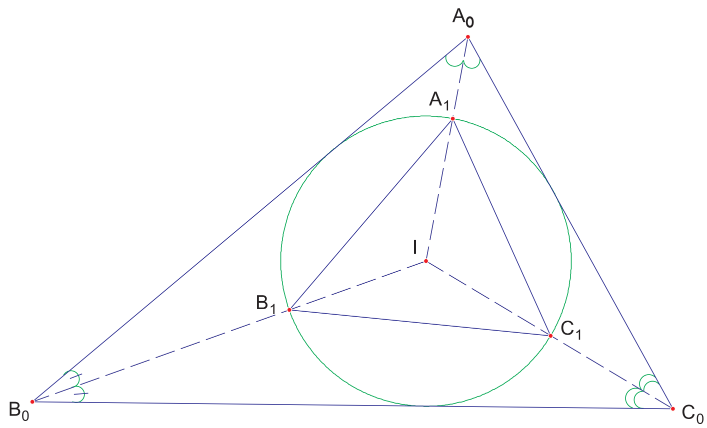

FIGURE 4. *A second construction using the incircle*

A similar problem with the one above appeared in [10].

THEOREM 3 (ANOTHER INCIRCLE CONSTRUCTION). *Given an arbitrary triangle $T_0 = \triangle A_0B_0C_0$ with incenter I denote by A_1, B_1, C_1 the points where the line segments A_0I, B_0I and C_0I respectively intersect the incircle of T_0. Define the new triangle $T_1 = \triangle A_1B_1C_1$ (see Figure 4). In the same way, we can construct $T_2 = \triangle A_2B_2C_2, \cdots$, $T_n = \triangle A_nB_nC_n \cdots$. The limiting shape of the triangle sequence $\{T_n\}_{n\geq 0}$, as $n \to \infty$ is an equilateral triangle.*

PROOF. (Sketch)

As in the previous problem there is a simple linear dependence between the angles of T_1 and those of T_0.

$$\begin{bmatrix} \widehat{A_1} \\ \widehat{B_1} \\ \widehat{C_1} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} \widehat{A_0} \\ \widehat{B_0} \\ \widehat{C_0} \end{bmatrix}, \quad \text{which again by induction gives that}$$

$$\lim_{n\to\infty} \begin{bmatrix} \widehat{A}_n \\ \widehat{B}_n \\ \widehat{C}_n \end{bmatrix} = \lim_{n\to\infty} \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}^n \cdot \begin{bmatrix} \widehat{A}_0 \\ \widehat{B}_0 \\ \widehat{C}_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} \widehat{A}_0 \\ \widehat{B}_0 \\ \widehat{C}_0 \end{bmatrix} = \begin{bmatrix} \pi/3 \\ \pi/3 \\ \pi/3 \end{bmatrix}.$$

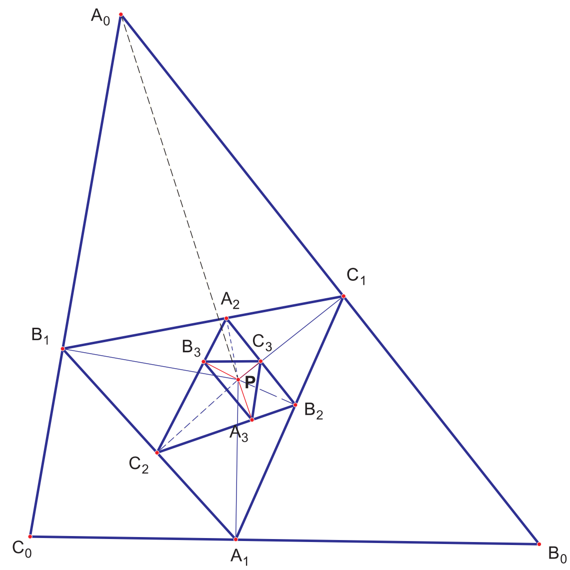

FIGURE 5. *Neuberg's construction*

As in the previous case, the limiting shape is an equilateral triangle. Moreover, it can be easily shown that $\{T_n\}_{n\geq 0}$ converges to a point but again, we are unable to express the position of this limiting point in finite terms. $\square$

The next result is attributed to Neuberg – see e.g. [5, 14, 16].

THEOREM 4 (NEUBERG). *Let $T_0 = \triangle A_0B_0C_0$ be an arbitrary triangle and let P be a point inside the triangle. Drop the perpendiculars from the point P onto the lines, A_0B_0, A_0C_0 and B_0C_0. We label the points of intersection C_1, B_1 and A_1, respectively. We can now form a new triangle, $T_1 = \triangle A_1B_1C_1$. Similarly, we drop the perpendiculars from P onto A_1B_1, A_1C_1 and B_1C_1. We label the points of intersection C_2, B_2 and A_2, respectively. Thus, we can form $T_2 = \triangle A_2B_2C_2$ – see Figure 5. Finally, we construct triangle $T_3 = \triangle A_3B_3C_3$ in a similar manner.*

Then, triangle $\triangle A_0B_0C_0$ is similar to triangle $\triangle A_3B_3C_3$.

We include the short proof.

PROOF. Draw the lines from A_0 to P, B_0 to P and C_0 to P. P lies on the circumcircles of the following triangles: $\triangle A_0B_1C_1$, $\triangle A_2B_1C_2$, $\triangle A_3B_3C_2$, $\triangle A_2B_2C_1$, and $\triangle A_3B_2C_3$. Thus we have

$$\widehat{C_1A_0P} = \widehat{C_1B_1P} = \widehat{A_2B_1P} = \widehat{A_2C_2P} = \widehat{B_3C_2P} = \widehat{B_3A_3P}$$

and

$$\widehat{PA_0B_1} = \widehat{PC_1B_1} = \widehat{PC_1A_2} = \widehat{PB_2A_2} = \widehat{PB_2C_3} = \widehat{PA_3C_3}$$

Thus

$$\widehat{A_0} = \widehat{C_1A_0P} + \widehat{PA_0B_1} = \widehat{B_3A_3P} + \widehat{PA_3C_3} = \widehat{A_3}.$$

Similarly, $\widehat{B_0} = \widehat{B_3}$. Therefore by angle-angle, $\triangle A_0B_0C_0$ and $\triangle A_3B_3C_3$ are similar. $\square$

Corollary. Suppose we continue constructing the triangles T_4, T_5, ... in the manner described in the statement of the Theorem 4. Then, the above proof implies that the shape sequence of $\{T_n\}_{n\geq 0}$ is periodical of period 3.

Let us note that depending on the shape of the initial triangle as well as the position of point P the triangle sequence in the problem above must not be necessarily nested.

The following problem has been studied in [12].

Definition. The *pedal triangle* of a given triangle $\triangle ABC$ is the triangle whose vertices are the feet of altitudes from A, B and C.

It is therefore natural to consider

The pedal triangle construction: Given any triangle $T_0 = \triangle A_0B_0C_0$ define $T_1 = \triangle A_1B_1C_1$ to be the pedal triangle of T_1 – see Figure 6.

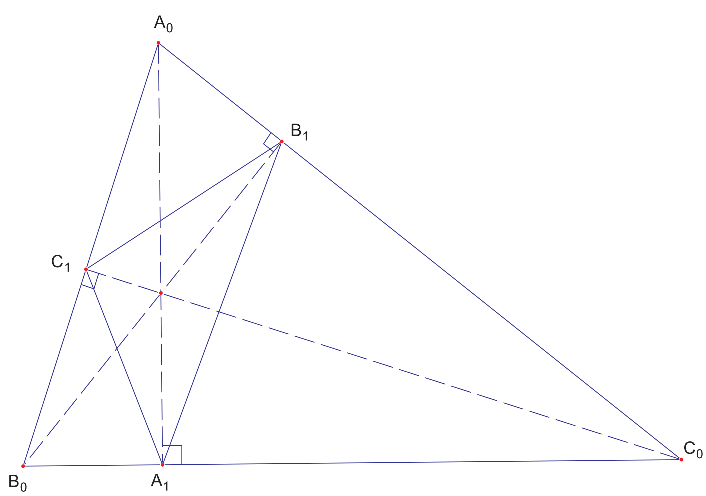

Figure 6. *Pedal triangle*

Iterating this procedure, we define T_{n+1} to be the pedal triangle of T_n. Note that this is not necessarily a nested triangle sequence. The same questions regarding the limiting behavior of $\{T_n\}_{n\geq 0}$ can be asked.

Kingston and Synge found in [12] necessary and sufficient conditions for the shape-sequence of $\{T_n\}_{n\geq 0}$ to be periodical for any given period, p. Moreover, they showed that there are triangles T_0 for which the periodicity phenomenon appears only after an arbitrarily large number of iterations (they call this periodicity with delay).

In other words, they show that given any positive integers p and d, there is a choice for T_0 such that no two triangles in the list $\{T_0, T_1, \ldots, T_d\}$ are similar to each other but T_k is similar to T_{k+p} for every $k \geq d$. We thus encounter in this case a somewhat similar situation to the one mentioned in the fixed division ratio construction.

We finally got to the problem that represents the main goal of this paper.

What if the vertices of triangle T_{n+1} are the feet of the *angle bisectors* of T_n for every $n \geq 0$? Trimble [15] has shown that if T_0 is *isosceles* then the limiting shape is that of an equilateral triangle. No proof has been published for the case when T_0 is an arbitrary triangle. We present such a proof in the next section.

3. Triangles formed by angle bisectors

PROBLEM. Let T_0 be an arbitrary triangle with vertices A_0, B_0 and C_0, and let T_1 be the triangle formed by the intersection points of the angle bisectors of T_0 on its three sides (see Figure 7). Construct $T_2, T_3, \cdots$ in the same manner.

We will prove the following

THEOREM 5. *The sequence $\{T_n\}_{n\geq 0}$ converges (in shape) to an equilateral triangle.*

OBSERVATION. As mentioned above, Trimble [15] proved the above theorem for the special case when T_0 is an isosceles triangle. We will present a simpler proof of this particular case. The main difficulty in proving the general statement is that there is no linear recurrence relationship between the angles of T_n and the angles of T_{n+1} (as it happened in Theorems 2 and 3, for instance).

The proof of Theorem 5 will consist of a sequence of lemmata – some of which are rather computationally involved. We used MAPLE to perform and check these calculations.

NOTATION. Let a_n, b_n, c_n denote the lengths of the sides B_nC_n, C_nA_n and A_nB_n respectively.

We will first express c_{n+1} in terms of a_n, b_n and c_n.

By the angle bisector theorem in triangle $A_nB_nC_n$ we have that

$$\frac{B_nA_{n+1}}{A_{n+1}C_n} = \frac{A_nB_n}{A_nC_n} = \frac{c_n}{b_n}$$

from which by using derived proportions we obtain

$$C_nA_{n+1} = \frac{a_nb_n}{b_n + c_n}. \tag{1}$$

Similarly, it can be shown that

$$C_nB_{n+1} = \frac{a_nb_n}{a_n + c_n}. \tag{2}$$

Using now the cosine rule in $\triangle A_{n+1}B_{n+1}C_n$ we have that

$$A_{n+1}B_{n+1}^2 = C_nA_{n+1}^2 + C_nB_{n+1}^2 - 2 \cdot C_nA_{n+1} \cdot C_nB_{n+1} \cdot \cos\widehat{C_n}$$

which after using the notation introduced above as well as (1) and (2) becomes

$$c_{n+1}^2 = \frac{a_n^2b_n^2}{(a_n+c_n)^2} + \frac{a_n^2b_n^2}{(b_n+c_n)^2} - 2\frac{a_n^2b_n^2}{(a_n+c_n)(b_n+c_n)}\cos\widehat{C_n}.$$

But if we use the cosine rule in $\triangle A_nB_nC_n$, we get

$$\cos\widehat{C_n} = \frac{b_n^2 + a_n^2 - c_n^2}{2a_nb_n}.$$

Thus

$$c_{n+1}^2 = \frac{a_n^2b_n^2}{(a_n+c_n)^2} + \frac{a_n^2b_n^2}{(b_n+c_n)^2} - \frac{a_nb_n\left(b_n^2 + a_n^2 - c_n^2\right)}{(a_n+c_n)(b_n+c_n)}.$$

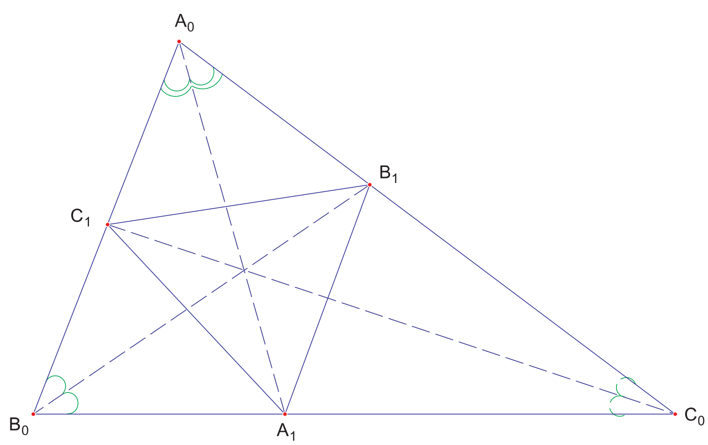

FIGURE 7. *T_1 is determined by the feet of the angle bisectors of T_0*

Similarly,

$$a_{n+1}^2 = \frac{b_n^2 c_n^2}{(a_n+b_n)^2} + \frac{b_n^2 c_n^2}{(a_n+c_n)^2} - \frac{b_n c_n \left(b_n^2+c_n^2-a_n^2\right)}{(a_n+b_n)(a_n+c_n)} \tag{3}$$

and

$$b_{n+1}^2 = \frac{c_n^2 a_n^2}{(b_n+c_n)^2} + \frac{c_n^2 a_n^2}{(a_n+b_n)^2} - \frac{c_n a_n \left(c_n^2+a_n^2-b_n^2\right)}{(b_n+c_n)(a_n+b_n)}.$$

Observation. The recurrence relations above do not seem to be particularly simple. However, if the initial triangle T_0 is isosceles the problem is readily solved.

Theorem 6 (The Isosceles Case). *If we repeatedly take the angle bisectors of an isosceles triangle, the resulting sequence of triangles* $\{\triangle A_n B_n C_n\}_{n=0}^{\infty}$, *converges in shape towards an equilateral triangle.*

Proof. Let us suppose that T_0 is isosceles; assume for instance that $b_0 = c_0$. Using the recurrence relations (3), a straightforward induction reasoning implies that $b_n = c_n$ for all $n \geq 0$.

Then, using (3) we can rewrite a_{n+1}^2 and b_{n+1}^2 as follows

$$a_{n+1}^2 = \frac{a_n^2\, b_n^2}{(a_n+b_n)^2}$$

and

$$b_{n+1}^2 = \frac{a_n^2\, \left(5b_n^2 - a_n^2\right)}{4(a_n+b_n)^2}.$$

Combining the last two equalities we obtain that

$$\frac{a_{n+1}^2}{b_{n+1}^2} = \frac{4b_n^2}{5b_n^2 - a_n^2}.$$

Obviously, it would suffice to show that the ratio $d_n := a_n^2/b_n^2$ tends to 1 as n approaches infinity. The last equation can be written as

$$d_{n+1} = \frac{4}{5-d_n} \tag{4}$$

where $0 < d_0 < 4$, the last inequality being a consequence of the triangle inequality $a_0 < b_0 + c_0 = 2\,b_0$.

Obviously, for every $n \geq 0$ we have $0 < d_n < 4$. Moreover, it can be easily shown that $\{d_n\}_{n\geq 0}$ is *monotonic* (strictly increasing if $d_0 < 1$, strictly decreasing if $d_0 > 1$ and constant if $d_0 = 1$). This can be shown using the fact that $d_{n+1} = f(d_n)$ where $f(x) = 4/(5-x)$. We skip the simple induction in favor of presenting a "picture proof" – see Figure 8.

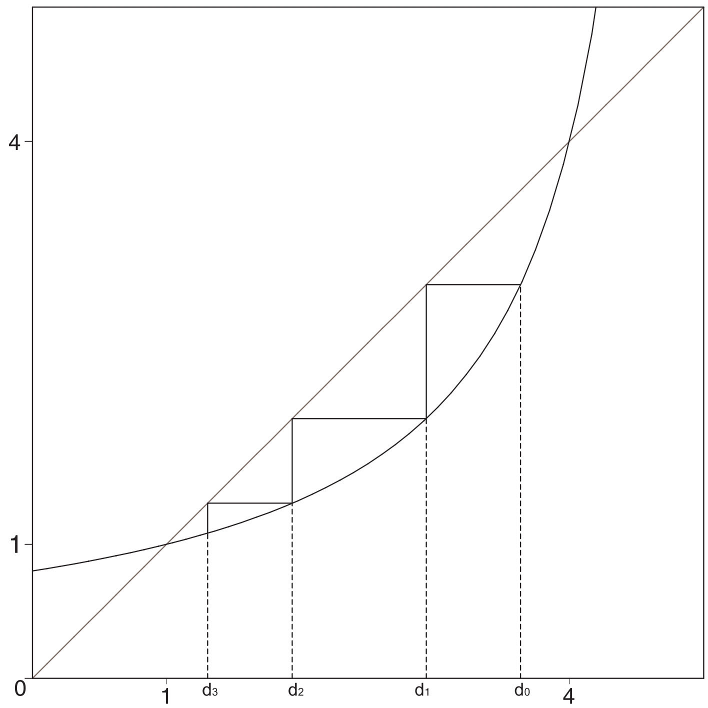

FIGURE 8. $\{d_n\}_{n\geq 0}$ *is decreasing and convergent to* 1 *when* $d_0 > 1$

Therefore, $\{d_n\}_{n\geq 0}$ is bounded and monotonic, and therefore convergent. Passing to the limit in the recurrence relationship (4) we have that

$$L = \frac{4}{5-L} \quad \text{where} \quad L := \lim_{n\to\infty} d_n$$

from which $L^2 - 5L + 4 = (L-1)(L-4) = 0$. But $L \neq 4$ since from the Figure 8 above we can easily see that $d_n \leq \max\{1, d_0\} < 4$. Therefore, $L = 1$ and the proof is complete. □

We now return to the case when $\triangle A_0B_0C_0$ is an arbitrary triangle. We start with the following

LEMMA 1. *If* $c_0 = \max\{a_0, b_0, c_0\}$ *then* $c_n = \max\{a_n, b_n, c_n\}$ *for every* $n \geq 0$.

PROOF. Induction on n. Consider the difference $c_{n+1}^2 - a_{n+1}^2$. From (3) after some simplifications it follows that

$$\begin{aligned} & c_{n+1}^2 - a_{n+1}^2 \\ & = \frac{a_n\, b_n\, c_n\, (c_n - a_n) \cdot \left[(c_n^2 - b_n^2)(a_n + b_n + c_n) + a_n(a_n + b_n)(a_n + c_n) \right]}{(a_n + b_n)^2 (b_n + c_n)^2 (a_n + c_n)} \end{aligned} \tag{5}$$

Since by the induction hypothesis $c_n \geq a_n$ and $c_n \geq b_n$, every factor in the right hand side of the above equation is nonnegative. Hence, $c_{n+1} \geq a_{n+1}$. Analogously, it can be shown that $c_{n+1} \geq b_{n+1}$. This ends the induction proof.

Notation. Define the sequences $\{r_n\}_{n \geq 0}$ and $\{s_n\}_{n \geq 0}$ by

$$r_n := \frac{a_n^2}{c_n^2}; \qquad s_n := \frac{b_n^2}{c_n^2}.$$

Clearly, by the above lemma we have that $0 < r_n, s_n \leq 1$ for all $n \geq 0$. Obviously, for proving Theorem 5 it would suffice to show that these two sequences converge to 1. However, at this point it is not clear whether $\{r_n\}_{n \geq 0}$ and $\{s_n\}_{n \geq 0}$ are convergent. Moreover, these sequences need *not be monotonic*, fact which makes our proof more laborious.

We need two technical results.

Lemma 2. *Let* $t_n = (r_n - s_n)^2$. *Then* $\lim_{n \to \infty} t_n = 0$.

Proof. Clearly, $0 \leq t_n < 1$ for every $n \geq 0$. We will show that

$$t_{n+1} \leq \frac{t_n + t_n^2}{2} \quad \text{for every} \quad n \geq 0. \tag{6}$$

For now, assume (6) holds true. Then we obtain

$$t_{n+1} \leq \frac{t_n + t_n^2}{2} \leq \frac{t_n + t_n}{2} = t_n$$

that is, $\{t_n\}_{n \geq 0}$ is decreasing and bounded, therefore it converges to some limit $l \in [0, 1]$. If we pass to the limit in (6) it follows that

$$\lim_{n \to \infty} t_{n+1} \leq \lim_{n \to \infty} \frac{t_n + t_n^2}{2} \implies l \leq \frac{l + l^2}{2}.$$

So,

$$l(1 - l) \leq 0 \implies l = 0 \text{ or } l = 1.$$

But $l \neq 1$, or else $t_0 = 1$ since the sequence is decreasing. But this means that $(r_0 - s_0)^2 = 1$, that is, either r_0 or s_0 is equal to 0, which is impossible since we started with a non-degenerate triangle T_0. Thus $l = 0$ and therefore $|r_n - s_n| \to 0$ as $n \to \infty$.

It remains to prove inequality (6). Denote $a_n = c_n - u$ and $b_n = c_n - v$. Since c_n is greater than both a_n and b_n it follows that $u, v \geq 0$. By the triangle inequality, $a_n + b_n - c_n = w > 0$. Hence,

$$a_n = v + w, b_n = u + w \quad \text{and} \quad c_n = u + v + w. \tag{7}$$

Substituting now (7) into the expression $\frac{t_n + t_n^2}{2} - t_{n+1}$ we obtain that

$$\frac{t_n + t_n^2}{2} - t_{n+1} = \frac{P_1(u, v, w)}{Q_1(u, v, w)} \tag{8}$$

where $P_1(u, v, w)$ and $Q_1(u, v, w)$ are polynomials of degree 16 in the nonnegative variables u, v and w taking only nonnegative values. Hence, $\frac{t_n + t_n^2}{2} - t_{n+1} \geq 0$ which proves (6) and with it the entire Lemma 2. $\square$

The second intermediate result we need is given in the following

Lemma 3. *Let $x_n = \min\{r_n, s_n\} = \frac{1}{2} \cdot (r_n + s_n - |r_n - s_n|)$. Then the sequence $\{x_n\}_{n \geq 0}$ is convergent.*

Proof. Obviously, $0 < x_n \leq 1$ for all $n \geq 0$ so the sequence is bounded. We will show that $\{x_n\}_{n \geq 0}$ is increasing from which the result stated above will follow.

Without loss of generality suppose that for a given n, $x_n = \min\{r_n, s_n\} = r_n$. We want to show that $x_{n+1} \geq x_n$, which is equivalent in this case to proving that $r_{n+1} \geq r_n$ and $s_{n+1} \geq r_n$.

Let us use the same notations (7) from the previous lemma.

It is easily shown that $r_n - s_n = (v - u)(v + u + 2w)/(u + v + w)^2$. The assumption that $r_n \leq s_n$, implies that $v \leq u$.

Let us denote $u = v + z$, where $z \geq 0$. Then equalities in (7) become

$$a_n = v + w, b_n = v + w + z \quad \text{and} \quad c_n = 2v + w + z. \tag{9}$$

Using now (9), a straightforward Maple computation gives that

$$r_{n+1} - r_n = \frac{P_2(v, w, z)}{Q_2(v, w, z)} \geq 0 \tag{10}$$

since both $P_2(v, w, z)$ and $Q_2(v, w, z)$ are polynomials of degree 7 in the nonnegative variables v, w and z having all coefficients positive.

Similarly, using again (9), we obtain that

$$s_{n+1} - r_n = \frac{P_3(v, w, z)}{Q_3(v, w, z)} \geq 0 \tag{11}$$

since $P_3(v, w, z)$ and $Q_3(v, w, z)$ are polynomials of degree 7 in the nonnegative variables v, w and z having all coefficients positive.

It follows that if $x_n = r_n$ then $x_{n+1} \geq x_n$. The case when $x_n = s_n$ is treated similarly. This finishes the proof of lemma 3. $\square$

We are now in position to prove Theorem 5.

By lemmata 2 and 3, we have that both sequences $\{r_n + s_n\}_{n\geq 0}$ and $\{r_n - s_n\}_{n\geq 0}$ do converge and the second one converges to 0. Therefore both sequences $\{r_n\}$ and $\{s_n\}$ converge to the same limit and hence the sequences defined by the general terms

$$R_n = \sqrt{r_n} = a_n/c_n \quad \text{and} \quad S_n = \sqrt{s_n} = b_n/c_n$$

converge to a common limit, Λ. Recall that we would like to show that $\Lambda = 1$.

Using the equations from (3) we derive the following equality

$$\begin{aligned} &R_{n+1}^2 \qquad (12)\\ &= \frac{-(S_n+1)^2(-S_n^3 - R_nS_n^2 + S_n^2 + 3R_nS_n + S_n + R_n^2S_n - 1 + R_n^3 + R_n^2 - R_n)}{(R_n^3 + R_n^2 - R_n^2S_n - 3R_nS_n - R_nS_n^2 - R_n + S_n^2 - S_n + S_n^3 - 1)(R_n + S_n)^2}. \end{aligned}$$

We know that $\lim_{n\to\infty} R_n = \lim_{n\to\infty} S_n = \Lambda$ so by passing to the limit in (12) we get that

$$\Lambda^2 = \frac{\left(5\Lambda^2 - 1\right)(1+\Lambda)^2}{4\Lambda^2(1+\Lambda)^2} = \frac{5\Lambda^2 - 1}{4\Lambda^2}$$

which readily implies that either $\Lambda = 1/2$ or $\Lambda = 1$.

We still have to eliminate the first possibility. Notice that if $\Lambda = 1/2$ then the limiting triangle would be a flat isosceles triangle.

Consider the following sequence

$$g_n := \frac{16\,A_n^2}{c_n^4}$$

where A_n denotes the area of the n^{th} triangle and c_n is the side of maximum length (see Lemma 1).

From Heron's formula is easy to deduce that

$$16\,A_n^2 = -a_n^4 - b_n^4 - c_n^4 + 2a_n^2\,b_n^2 + 2a_n^2\,c_n^2 + 2b_n^2\,c_n^2$$

which immediately implies that

$$g_n = -r_n^2 - s_n^2 - 1 + 2r_n\,s_n + 2r_n + 2s_n. \qquad (13)$$

Clearly, since $\{r_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$ are convergent then the sequence $\{g_n\}_{n\geq 0}$ is convergent, too. Notice that if $\Lambda = 1/2$ then $\lim_{n\to\infty} g_n = 0$. However, we will prove below that this is impossible.

The proof of Theorem 5 will be finished as soon as we manage to show

LEMMA 4. *The sequence $\{g_n\}_{n\geq 0}$ is increasing.*

PROOF. We will use the same approach as in Lemma 2. Denote $a_n = c_n - u$ and $b_n = c_n - v$. Since c_n is greater than both a_n and b_n it follows that $u, v \geq 0$. By the triangle inequality, $a_n + b_n - c_n = w > 0$. Hence, $a_n = v + w$, $b_n = u + w$ and $c_n = u + v + w$. Substituting now a_n, b_n, c_n into the expression of g_n we obtain that

$$g_{n+1} - g_n = \frac{P_4(u,v,w)}{Q_4(u,v,w)} \tag{14}$$

where $P_4(u,v,w)$ and $Q_4(u,v,w)$ are polynomials of degree 12 in the nonnegative variables u, v, w having all coefficients positive. It follows the $\{g_n\}_{n\geq 0}$ is an increasing sequence. □

Since $\{g_n\}$ is increasing and has only positive terms it cannot converge to 0. This means that $\Lambda = 1$ therefore the limiting triangle is equilateral. The proof of Theorem 5 is now complete.

4. Open problems

It seems that finding the *limiting point* of a certain iterative geometric process is a much more difficult problem than the one concerning the *limiting shape*. In particular, we would be interested in finding these limiting points in the cases of the incircle-circumcircle sequence (the Poncelet point that is) and in the case of the last construction (triangles determined by the angle bisectors). Also, it would be interesting to find all values of the division ratio $0 < s < 1$ for which the triangle sequence constructed as in Theorem 1 is divergent (in shape).

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