Properties of bisect-diagonal quadrilaterals

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1. *Introduction*

The general class of quadrilaterals where one diagonal is bisected by the other diagonal has appeared very rarely in the geometrical literature, but they have been named several times in connection with quadrilateral classifications. Günter Graumann strangely gave these objects two different names in [1, pp. 192, 194]: sloping-kite and sliding-kite. A. Ramachandran called them slant kites in [2, p. 54] and Michael de Villiers called them bisecting quadrilaterals in [3, pp. 19, 206]. The latter is a pretty good name, although a bit confusing: what exactly is bisected?

We have found no papers and only two books where any theorems on such quadrilaterals are studied. In each of the books, one necessary and sufficient condition for such quadrilaterals is proved (see Theorem 1 and 2 in the next section). The purpose of this paper is to investigate basic properties of *convex* bisecting quadrilaterals, but we have chosen to give them a slightly different name. Let us first remind the reader that a quadrilateral whose diagonals have equal lengths is called an equidiagonal quadrilateral and one whose diagonals are perpendicular is called an orthodiagonal quadrilateral. In comparison with these names, we will make the following definition:

Definition: A *bisect-diagonal quadrilateral* is a quadrilateral where at least one diagonal is bisected by the other diagonal (see Figure 1).

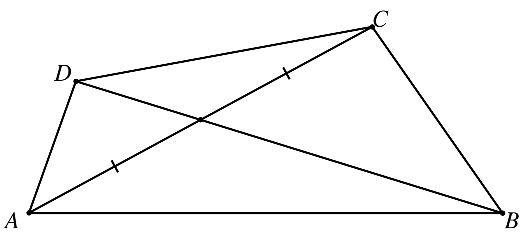

FIGURE 1

An important special case of bisect-diagonal quadrilaterals are the kites, which at the same time are both bisect-diagonal and orthodiagonal. Other special cases are all parallelograms, where both diagonals are bisected by each other, and their subsets: rhombi, rectangles, and squares.

2. *Subtriangles in a quadrilateral*

We will use the following notations in this section. In a convex quadrilateral *ABCD* where the diagonals intersect at *P*, let the areas of the triangles *ABD*, *BCD*, *CDB*, *DAC* be T_A, T_B, T_C, T_D respectively, and the areas

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of the triangles *ABP*, *BCP*, *CDP*, *DAP* be T_a, T_b, T_c, T_d respectively. Then trivially we have

$$T_A + T_C = T_B + T_D.$$

Let's start with a simple characteristic property of bisect-diagonal quadrilaterals that was proved more or less in the same way in [4, pp. 54-55].

Theorem 1: A diagonal in a convex quadrilateral bisects the other diagonal if, and only if, it also bisects the area of the quadrilateral.

Proof: Let h_1 and h_2 be the heights in the triangles *ABD* and *CBD* respectively to their common side *BD* (see Figure 2(a)). By similar triangles $|AP| = |CP|$ if, and only if, $h_1 = h_2$, that is, if, and only if, $T_A = T_C$.

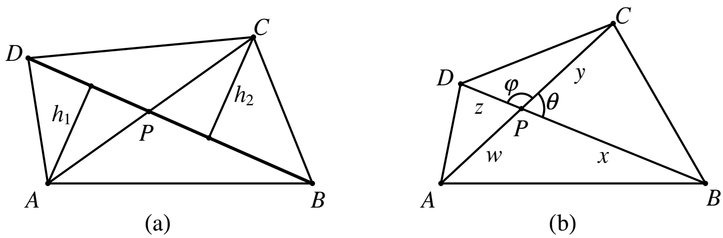

FIGURE 2: Heights to a diagonal (a) and the diagonal parts (b)

To prove the next necessary and sufficient condition for a bisect-diagonal quadrilateral was a problem proposed by Titu Andreescu in 2001 for the Korean Mathematics Competition, according to [5, p. 189], which gave another proof to ours. The problem was formulated slightly differently and did not use any name for this class of quadrilaterals.

Theorem 2: *ABCD* is a bisect-diagonal quadrilateral if, and only if,

$$T_a + T_c = T_b + T_d.$$

Proof: Let $AP = w$, $BP = x$, $CP = y$, $DP = z$ and let the two angles between the diagonals be θ and φ (see Figure 2(b)). Then $\sin\varphi = \sin\theta \neq 0$. The following statements are equivalent:

$$T_a + T_c = T_b + T_d$$

$$\tfrac{1}{2}wx \sin\varphi + \tfrac{1}{2}yz \sin\varphi = \tfrac{1}{2}xy \sin\theta + \tfrac{1}{2}zw \sin\theta$$

$$wx + yz = xy + zw$$

$$(w - y)(x - z) = 0$$

from which the result follows.

Corollary: *ABCD* is a bisect-diagonal quadrilateral if, and only if,

$$\frac{1}{T_a} + \frac{1}{T_c} = \frac{1}{T_b} + \frac{1}{T_d}.$$

Proof: Immediate from Theorem 2, since in all convex quadrilaterals we have

$$T_a T_c = \tfrac{1}{4} wxyz \sin^2 \theta = T_b T_d.$$

3. *A symmetry between diagonals and bimedians*

A *bimedian* is a line segment connecting the midpoints of two opposite sides of a quadrilateral. In this section we shall study when the two bimedians intersect on a diagonal and vice versa.

Theorem 3: The bimedians in a convex quadrilateral intersect on a diagonal if, and only if, it is a bisect-diagonal quadrilateral.

Proof: A property that all convex quadrilaterals have is that the two bimedians and the line segment that connects the midpoints of the diagonals are concurrent. A short proof was given in [4, p. 54] and [6, pp. 108-109]. From this property it is evident that the bimedians intersect on the diagonal that is the bisecting one in a bisect-diagonal quadrilateral (see Figure 3).

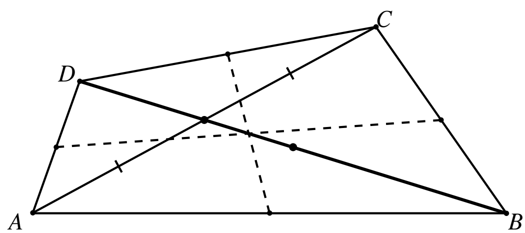

FIGURE 3: The bimedians in a bisect-diagonal quadrilateral

Conversely, if we know that the bimedians intersect on one diagonal in a convex quadrilateral, and of course that diagonal has its midpoint on itself, then the midpoint of the other diagonal must also lie on this diagonal since there is only one straight line through two points. This proves that it is a bisect-diagonal quadrilateral.

The following theorem where the bimedians and the diagonals have changed roles was proved as Theorem 15 in [7], where it was stated differently:

Theorem 4: The diagonals in a convex quadrilateral intersect on a bimedian if, and only if, it is a trapezium.

4. *The special cases kite and parallelogram*

In the remaining part of this paper, the sides of a bisect-diagonal quadrilateral $ABCD$ will be denoted $AB = a$, $BC = b$, $CD = c$, $DA = d$, and the diagonals $AC = p$ and $BD = q$.

Theorem 5: In a bisect-diagonal quadrilateral, the two angles opposite the bisecting diagonal are equal if, and only if, the quadrilateral is either a kite or a parallelogram.

Proof: Suppose diagonal p is bisected by diagonal q, and $\angle BAD = \alpha = \angle DCB$. By Theorem 1, we have $\frac{1}{2}ad \sin\alpha = \frac{1}{2}bc \sin\alpha$, so $ad = bc$. By the cosine rule,

$$a^2 + d^2 - 2ad\cos\alpha = q^2 = b^2 + c^2 - 2bc\cos\alpha.$$

Thus $a^2 + d^2 + 2ad = b^2 + c^2 + 2bc$, that is $a + d = b + c$. Multiplying through by a, we get $(a - b)(a - c) = 0$, so that $a = b$ and we have a kite, or $a = c$ and we have a parallelogram. The converse is clear.

To prove the next theorem, we shall need the following triangle property.

Lemma: In triangle UVW, let X be the foot of the altitude through U and let Y be the midpoint of VW. Assume, without loss of generality, that $|UV| \geqslant |UW|$. Then

$$|UV|^2 - |UW|^2 = |XV|^2 - |XW|^2 = 2|VW||XY|.$$

The proof is immediate by the Pythagorean theorem.

Theorem 6: If diagonal p is bisected by diagonal q, then $ABCD$ is a parallelogram if, and only if,

$$a^2 + b^2 = c^2 + d^2;$$

$ABCD$ is a kite if, and only if,

$$a^2 + c^2 = b^2 + d^2;$$

and $ABCD$ is either a kite or a parallelogram if, and only if,

$$a^2 + d^2 = b^2 + c^2.$$

Proof: Let M and N be the midpoints of AC and BD, where M is also on BD. If $a^2 - d^2 = c^2 - b^2$, then the feet of the perpendiculars from A and C onto BD are equidistant from N by the lemma. But they are also equidistant from M, so $M = N$ and we have a parallelogram.

If $a^2 - b^2 = d^2 - c^2$, then the feet of the perpendiculars from B and D onto AC coincide, that is, $AC \perp BD$, and we have a kite.

If $a^2 - b^2 = c^2 - d^2$, then the feet of the perpendiculars from B and D onto AC are equidistant from M, so either they both coincide with M and we have a kite, or else by congruent triangles $|BM| = |MD|$ and we have a parallelogram.

The converses are trivial.

5. *Metric relations in bisect-diagonal quadrilaterals*

There are two cases of bisect-diagonal quadrilaterals depending on whether diagonal p is bisected by diagonal q or vice versa. We only study one case, but the other only involves making the change of letters $b \leftrightarrow d$ in the metric relations.

Theorem 7: Diagonal p is bisected by diagonal q if, and only if, it has length

$$p = \frac{\sqrt{4q^2(a^2 + b^2 + c^2 + d^2 - q^2) - (a^2 + b^2 - c^2 - d^2)^2}}{2q}.$$

Proof: Let M be the midpoint of AC. Then $BD \leqslant BM + MD$ according to the triangle inequality in triangle BMD, where equality holds if, and only if, M lies on diagonal BD (see Figure 4).

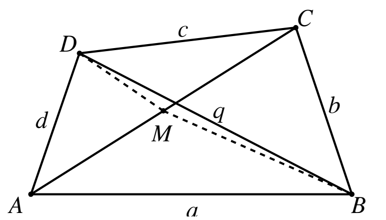

FIGURE 4: *ABCD* is bisect-diagonal if, and only if, *BDM* is degenerate

Applying Apollonius' theorem in triangles ABC and ADC, we have that p is bisected by q if, and only if,

$$q = \tfrac{1}{2}\sqrt{2(a^2 + b^2) - p^2} + \tfrac{1}{2}\sqrt{2(c^2 + d^2) - p^2}.$$

Squaring and then multiplying through by 2 yields

$$2q^2 - (a^2 + b^2 + c^2 + d^2 - p^2) = \sqrt{(2(a^2 + b^2) - p^2)(2(c^2 + d^2) - p^2)}.$$

Note that both sides are positive since the right-hand side is positive and equality holds. Squaring again and expanding a few (but not all) of the parentheses, we get after simplification

$$(a^2 + b^2 + c^2 + d^2)^2 - 4q^2(a^2 + b^2 + c^2 + d^2 - q^2) + 4p^2q^2 = 4(a^2 + b^2)(c^2 + d^2)$$

which is equivalent to

$$4p^2q^2 = 4q^2(a^2 + b^2 + c^2 + d^2 - q^2) - (a^2 + b^2 - c^2 - d^2)^2. \quad (1)$$

Now the formula for p follows at once from (1). We have equivalence in all steps of the derivation since we only square positive expressions and seek a positive solution.

Next we shall see that the lengths of the diagonals, the distance between the diagonal midpoints, and the area of a general bisect-diagonal quadrilateral can be expressed in terms of its four sides alone.

Theorem 8: If diagonal p is bisected by diagonal q and $ABCD$ is neither a kite nor a parallelogram, then

$$q = \sqrt{\frac{(a^2 - b^2 + c^2 - d^2)(a^2 + b^2 - c^2 - d^2)}{2(a^2 - b^2 - c^2 + d^2)}}$$

and

$$p = \sqrt{\frac{2(a^2 - b^2)^2(c^2 + d^2) - 2(a^2 + b^2)(c^2 - d^2)^2}{(a^2 - b^2 - c^2 + d^2)(a^2 - b^2 + c^2 - d^2)}}.$$

Proof: We apply Theorem 1 and a semi-factorised version of Heron's formula in triangles ABD and CBD. Equating the radicands yields

$$4a^2d^2 - (a^2 + d^2 - q^2)^2 = 4b^2c^2 - (b^2 + c^2 - q^2)^2.$$

Expanding the squares and solving for q^2, we get

$$2(a^2 - b^2 - c^2 + d^2)q^2 = (a^2 - d^2)^2 - (b^2 - c^2)^2 \qquad (2)$$

from which the first result follows.

The second formula is obtained by inserting the expression for q^2 from (2) into (1) and simplifying. We leave the details to the reader.

Theorem 9: In a bisect-diagonal quadrilateral that is not a kite, the distance v between the diagonal midpoints is given by

$$v = \sqrt{\frac{(a^2 - b^2 - c^2 + d^2)(a^2 + b^2 - c^2 - d^2)}{8(a^2 - b^2 + c^2 - d^2)}}.$$

Proof: In all convex quadrilaterals,

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4v^2 \qquad (3)$$

according to Euler's quadrilateral theorem [5, p. 14]. Inserting this into (1) yields

$$4p^2q^2 = 4q^2(p^2 + 4v^2) - (a^2 + b^2 - c^2 - d^2)^2.$$

The result follows after inserting the expression for q^2 and solving for v. The formula holds no matter which diagonal bisects the other since it is unaltered if b and d are interchanged.

A corollary is the well-known result that the diagonals bisect each other in a parallelogram.

The area of a bisect-diagonal quadrilateral is given by the following remarkable formula.

Theorem 10: If diagonal p is bisected by diagonal q and $ABCD$ is neither a kite nor a parallelogram, then the bisect-diagonal quadrilateral has area

$$K = \frac{Q_1 Q_2}{\left|a^2 - b^2 - c^2 + d^2\right|}$$

where

$$Q_1 = \sqrt{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)},$$

$$Q_2 = \sqrt{(a+b+c+d)(a-b-c+d)(a-b+c-d)(a+b-c-d)}.$$

Proof: Since, according to Theorem 1, diagonal q bisects the area, by applying a semi-factorised version of Heron's formula in triangle ABD, we get

$$K^2 = 4 \times \tfrac{1}{16}\left(4a^2d^2 - \left(a^2 + d^2 - q^2\right)^2\right).$$

Inserting the expression for q^2 from (2) and rewriting yields

$$16\left(a^2 - b^2 - c^2 + d^2\right)K^2 = 16a^2d^2\left(a^2 - b^2 - c^2 + d^2\right)^2 - u^2 \qquad (4)$$

where $u = 2\left(a^2 + d^2\right)\left(a^2 - b^2 - c^2 + d^2\right) - \left(a^2 - d^2\right)^2 + \left(b^2 - c^2\right)^2$ expands to

$$a^4 - 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4 + 6a^2d^2 - 2b^2d^2 - 2c^2d^2 + d^4.$$

The right-hand side of (4) is factorised into

$$\left(4ad\left(a^2 - b^2 - c^2 + d^2\right) + u\right)\left(4ad\left(a^2 - b^2 - c^2 + d^2\right) - u\right). \qquad (5)$$

The left parenthesis can be factorised by the fourth-degree binomial expression and some elementary algebra. We get

$$(a + d)^4 - 2\left(b^2 + c^2\right)(a + d)^2 + \left(b^2 - c^2\right)^2$$

$$= (a + d)^4 - (a + d)^2\left((b + c)^2 + (b - c)^2\right) + (b - c)^2(b + c)^2$$

$$= \left((a + d)^2 - (b - c)^2\right)\left((a + d)^2 - (b + c)^2\right)$$

$$= (a + d + b - c)(a + d - b + c)(a + d - b - c)(a + d + b + c).$$

In the same way we factorise the right parenthesis of (5). This yields

$$-(a - d + b - c)(a - d - b + c)(a - d - b - c)(a - d + b + c).$$

The result now follows from (4). Note that we rearranged the parentheses in the final factorisation when defining Q_1 and Q_2.

It is interesting to observe that both Q_1 and Q_2 stay the same if b and d are interchanged. In the case of a kite or a parallelogram, both Theorems 8 and 10 yield indeterminate quantities, as one would expect.

Let $S = \sqrt{(s - a)(s - b)(s - c)(s - d)}$, where $s = \frac{1}{2}(a + b + c + d)$ is the semi-perimeter, and note that $Q_1 = 4S$. Brahmagupta's formula says that S, or $\frac{1}{4}Q_1$, gives the area of the quadrilateral in the case where it is cyclic.

Corollary: Under the same assumptions as in Theorem 10, we have

$$K = S\sqrt{1 - 4\left(\frac{ad - bc}{a^2 - b^2 - c^2 + d^2}\right)^2}.$$

Proof: Rewriting Q_2^2 in the following way:

$$\begin{aligned}&(a + d + b + c)(a + d - b - c)(a - d + b - c)(a - d - b + c)\\ &= \left((a + d)^2 - (b + c)^2\right)\left((a - d)^2 - (b - c)^2\right)\\ &= \left(a^2 + d^2 - b^2 - c^2 + 2ad - 2bc\right)\left(a^2 + d^2 - b^2 - c^2 - 2ad + 2bc\right)\\ &= \left(a^2 + d^2 - b^2 - c^2\right)^2 - 4(ad - bc)^2\end{aligned}$$

we see that the formulas for K in this corollary and Theorem 10 are equivalent.

In the next section we shall see how this formula is related to cyclic bisect-diagonal quadrilaterals.

6. *Cyclic bisect-diagonal quadrilaterals*

A quadrilateral is called cyclic if all of its vertices lie on a circle. A kite is cyclic if, and only if, it is a right kite (a kite with a pair of opposite right angles), and a parallelogram is cyclic if, and only if, it is a rectangle. For the general bisect-diagonal quadrilateral we have the following beautiful result.

Theorem 11: A bisect-diagonal quadrilateral with consecutive sides a, b, c, d, which is neither a kite nor a parallelogram, is cyclic if, and only if,

$$\frac{a}{c} + \frac{c}{a} = \frac{b}{d} + \frac{d}{b}.$$

Proof: First note that the equation in the theorem holds if, and only if, $\frac{a}{c} = \frac{b}{d}$ or $\frac{a}{c} = \frac{d}{b}$, that is, if, and only if, $ad = bc$ or $ab = cd$. Suppose diagonal p is bisected by diagonal q, and let $\angle BAD = \alpha$ and $\angle DCB = \gamma$. By Theorem 1, $\frac{1}{2}ad\sin\alpha = \frac{1}{2}bc\sin\gamma$, so $ad = bc$ if, and only if, $\sin\alpha = \sin\gamma$. But by Theorem 5 we know $\alpha \neq \gamma$, so $ad = bc$ if, and only if, $\alpha + \gamma = \pi$, that is if, and only if, our quadrilateral is cyclic.

A similar argument holds if diagonal q is bisected by diagonal p, in which case $ab = cd$ if, and only if, the quadrilateral is cyclic.

There is a connection between Theorem 11 and the area formula in the corollary to Theorem 10. A quadrilateral is cyclic if, and only if, its area is given by Brahmagupta's formula. That is a consequence from a formula for the area of a convex quadrilateral (often known as Bretschneider's formula),

$$K = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd\cos^2\left(\frac{\alpha + \gamma}{2}\right)}$$

where s is the semi-perimeter. This formula is equivalent to Brahmagupta's formula if, and only if, $\alpha + \gamma = \pi$. The cyclic characterisation follows since $K = S$ in the corollary if, and only if, $ad = bc$ when diagonal p is bisected by diagonal q. By symmetry ($b \leftrightarrow d$), the quadrilateral is cyclic if, and only if, $ab = cd$ when diagonal q is bisected by diagonal p.

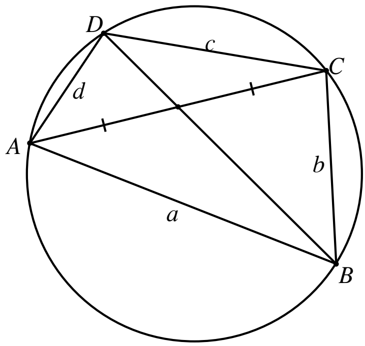

FIGURE 5: A cyclic bisect-diagonal quadrilateral

In cyclic quadrilaterals there are a lot of well-known metric relations and other properties. Here we shall only add to the existing collection of formulas two more that are different for cyclic bisect-diagonal quadrilaterals than for general cyclic quadrilaterals. The lengths of the diagonals in the latter case are:

$$p = \sqrt{\frac{(ac + bd)(ad + bc)}{ab + cd}}, \qquad q = \sqrt{\frac{(ab + cd)(ac + bd)}{ad + bc}}. \tag{6}$$

The formula for the bisecting diagonal can be simplified:

Theorem 12: Let $ABCD$ be cyclic, and let diagonal p be bisected by diagonal q. Then the latter has length

$$q = \sqrt{\frac{a^2 + b^2 + c^2 + d^2}{2}}.$$

Proof: From the proof of Theorem 11, we have $ad = bc$. Then from (6), we get

$$q^2 = \frac{a^2bc + adc^2 + adb^2 + bcd^2}{2ad} = \frac{ad\left(a^2 + b^2 + c^2 + d^2\right)}{2ad}.$$

Corollary: Let $ABCD$ be cyclic, and let diagonal p be bisected by diagonal q. Then the distance between the diagonal midpoints is

$$v = \tfrac{1}{2}\sqrt{q^2 - p^2}.$$

Proof: By Theorem 12, $a^2 + b^2 + c^2 + d^2 = 2q^2$, and the result follows immediately by putting this into Euler's quadrilateral theorem (3).

Of course, both diagonals bisect each other if, and only if, the quadrilateral is a cyclic parallelogram, which is equivalent to a rectangle.

7. *The bisect-diagonal quadrilateral in a classification*

Quadrilateral family trees have been studied for a long time, see [8] for a historical account and a new classification including what we considered to be the 18 most important classes of convex quadrilaterals at the time of writing that paper. Where does the bisect-diagonal quadrilateral fit into a classification of quadrilaterals? It turns out to be quite difficult to include it in such a tree if we want the tree to have some sort of symmetry. The problem arises when trying to answer the question: What is the symmetric quadrilateral to the trapezium? If using a side-angle symmetry (the most common one), the answer is the extangential quadrilateral (a quadrilateral with an excircle, see [8, p. 72]).

Another possibility is to focus on the diagonals, as has been done for instance by Ramachandran in [2, Table 1]. His classification was presented as a table, and just had properties of the diagonals as its scaffolding. There is however another way of interpreting his classification that is closely related to a pair of theorems we have included in this paper (Theorems 3 and 4). Due to the symmetric properties between the diagonals and the bimedians, Ramachandran's classification can be considered to be based on a diagonal-bimedian symmetry. To better get a grip of this, we have drawn a family tree of the 16 quadrilaterals Ramachandran included in his table, see Figure 6. There the bisect-diagonal quadrilateral is the symmetric partner to the trapezium.

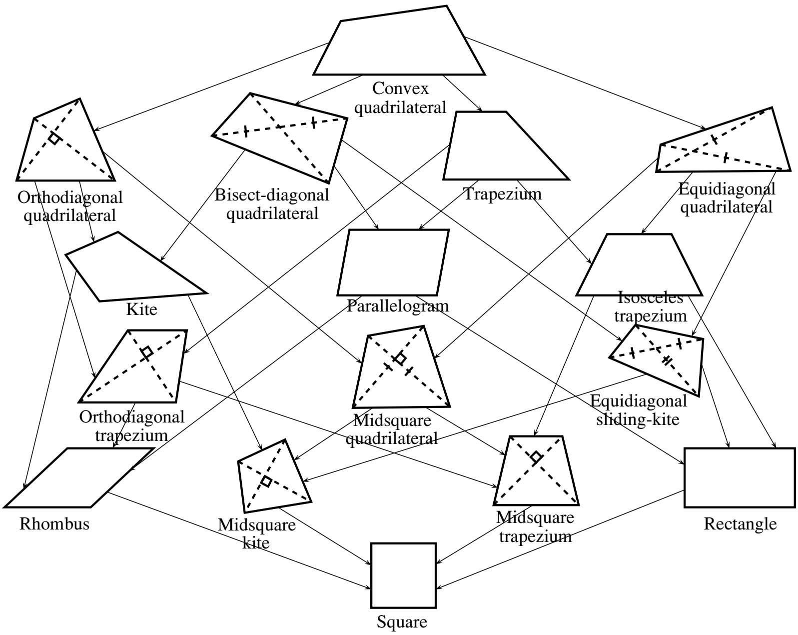

FIGURE 6: A quadrilateral family tree based on a diagonal-bimedian symmetry

There is another pair of theorems concerning the diagonal-bimedian symmetry that was proved as Theorem 7 in [9]. They state that the bimedians in a convex quadrilateral are equal if, and only if, the diagonals are perpendicular, and that the bimedians are perpendicular if and only if the diagonals are equal. The diagonal-bimedian symmetry in Figure 6 can be shown even more clearly as in Table 1 in the present paper (note that this is not the same as Table 1 in [2]).

The quadrilaterals whose names start with midsquare in Figure 6 and Table 1 were labelled so by us in [8] since the midpoints of their sides form a square. The name equidiagonal sliding-kite was also used by us in [8] when reviewing a classification due to Graumann, since he gave no name and it was what he called a sliding-kite with equal diagonals. Ramachandran referred to them as a slant kite with equal diagonals. They are bisect-equidiagonal quadrilaterals, but that is a rather cumbersome name.

No.	Quadrilateral	Characterisation
1	Orthodiagonal quadrilateral	Diagonals perpendicular
1′	Equidiagonal quadrilateral	Bimedians perpendicular
2	Bisect-diagonal quadrilateral	Bimedians meet on a diagonal
2′	Trapezium	Diagonals meet on a bimedian
3	Kite	1 + 2
3′	Isosceles trapezium	1′ + 2′
4 = 4′	Parallelogram	2 + 2′
5	Orthodiagonal trapezium	1 + 2′
5′	Equidiagonal sliding-kite	1′ + 2
6 = 6′	Midsquare quadrilateral	1 + 1′
7	Rhombus	3 + 4
7′	Rectangle	3′ + 4′
8	Midsquare kite	5′ + 6
8′	Midsquare trapezium	5 + 6′
9 = 9′	Square	7 + 7′ *or* 8 + 8′

TABLE 1: Diagonal or bimedian characterisations

We conclude with a challenge to the reader: Construct a classification of convex quadrilaterals that is built on some sort of symmetry which incorporate both the extangential quadrilateral and the bisect-diagonal quadrilateral, as well as all the other 17 important quadrilaterals in Figure 10 in [8] and whatever additional quadrilaterals are needed to achieve this symmetry. If you are successful, then please let us know.

Acknowledgement

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The answers to the *Nemo* page from March on proof were:

1.	William Blake	The Marriage of Heaven and Hell: Proverbs of Hell
2.	James Joyce	Ulysses Scylla and Charybdis
3.	Emily Dickinson	It troubled me as once I was
4.	Hanya Yanagihara	A Little Life: Chapter II
5.	William Shakespeare	All's Well that ends Well: Act 5 Scene 3
6	Jane Austen	Persuasion: Chapter 21

Congratulations to Martin Lukarewski and Henry Ricardo for identifying all of these quotations. The theme this month is algebra, its effects and its limitations. The quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd September 2017.

1. ‘Algebra, like laudanum, deadens pain,’ Fritz wrote. ‘But the study of algebra has confirmed for me that philosophy and mathematics, like mathematics and music, speak the same language.’
2. But, to see it, he must have overleaped at a bound the artificial barriers he had for many years been erecting, between himself and all those subtle essences of humanity which will elude the utmost cunning of algebra until the last trumpet ever to be sounded shall blow even algebra to wreck.

Continued on page 318.