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On the classification of convex quadrilaterals

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1. *Introduction*

We live in a golden age regarding the opportunities to explore Euclidean geometry. The access to dynamic geometry computer programs for everyone has made it very easy to study complex configurations in a way that has never been possible before. This is especially apparent in the study of triangle geometry, where the flow of new problems, properties, and papers is probably the highest it has ever been in the history of mathematics. Even though it has increased a bit in recent years, the interest in quadrilateral geometry is significantly lower. Why are triangles so much more popular than quadrilaterals? In fact, we think it would be more logical if the situation were reversed, since there are so many classes of quadrilaterals to explore. This is the primary reason we think that quadrilaterals are a lot more interesting to study than triangles.

For the last decade we have been especially interested in what characterises the different classes of convex quadrilaterals, that is, which properties in each class are unique to them. We refer to [1–8] where we proved many new necessary and sufficient conditions for bicentric quadrilaterals, tangential quadrilaterals, kites, orthodiagonal quadrilaterals, extangential quadrilaterals, trapezia*, equidiagonal quadrilaterals, and rectangles. A few of these classes had not been studied that much before, but we hope the interest in them will increase. It is common in school to learn about squares, rectangles, rhombi, parallelograms, trapezia, isosceles trapezia, and kites. However there are about a dozen other interesting classes of quadrilaterals that we think deserve more attention. At least their properties and characterisations can be explored in more advanced geometry courses, or they can make challenging assignments for gifted students within the regular curriculum.

In this paper we will first study the classification of quadrilaterals from a historical perspective and discuss the definition of a trapezium and an isosceles trapezium. Then we present our quadrilateral family tree built on the basis of inclusive definitions and symmetry that incorporate what we consider to be the 18 most important classes of convex quadrilaterals.

* Note that a trapezium (trapezia in plural) in British English is the same as a trapezoid in American English. Both of these names are used in the references, but to avoid confusion we use only the former here.

2. *From Euclid to de Villiers – previous classifications*

The story of classifying quadrilaterals begins with Euclid (3rd century BC). He gives the following definition, which is number 22 in book I of the *Elements* [9]:

> *Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.*

It is noteworthy that Euclid used trapezia for any 'irregular quadrilateral' other than the four he named. Another thing to note is that his definitions are exclusive. This means that he considers for instance rhombus and rhomboid (what we would call a parallelogram) to be totally different classes instead of thinking of a rhombus as a special case of a parallelogram. In the latter way of thinking, which is called an inclusive definition, we have that the rhombus is a subset of the class of parallelograms. This is the dominant way of making definitions in mathematics today. However there is a disturbing exception that we shall discuss in the next section.

The classification of quadrilaterals was extended by the Greek polymath Posidonius (1st century BC) and later used by Heron (1st century AD): see Figure 1 [10, p. 45]. More quadrilaterals have been added, but the dichotomous division is still used. This means that Posidonius defines a type of quadrilateral and then considers its negation. Thus he also used exclusive definitions.

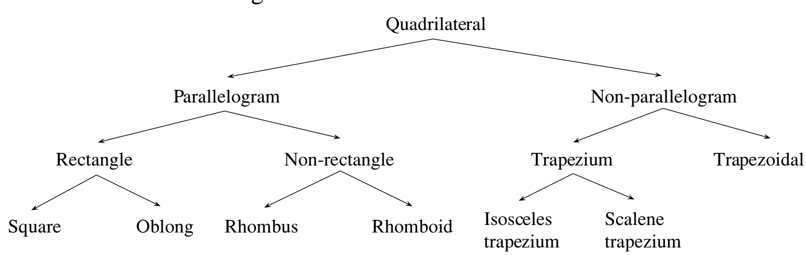

FIGURE 1: Posidonius' classification of quadrilaterals

The 9th century Indian mathematician Mahavira divided quadrilaterals into five classes: those with unequal sides (mostly *cyclic quadrilaterals* were considered, i.e. those with a circumcircle), all sides equal (rhombus and square), opposite sides equal (parallelogram and rectangle), two opposite sides equal (isosceles trapezium), and trapezia with three sides equal [11, p. 257] .

The French logician Petrus Ramus contributed in the 16th century with a classification that is very similar to that of Posidonius. He used the name oblique parallelogram instead of non-rectangle and also changed non-parallelogram to trapezium. However he did not subdivide the latter class any further [10, p. 46].

Nowadays there are two competing systems of classification for quadrilaterals: the exclusive and the inclusive. What is at an issue is whether the parallelograms shall be considered a subclass of the trapezia or not. Making a picture search on Google with the search words ‘quadrilateral classification’, it is evident that there are basically the following two family trees around today. Some small variations exist that we will discuss as well.

In a family tree based on the *exclusive* definition, the convex quadrilaterals are most often divided into trapezia, parallelograms, and kites. The trapezia are divided into isosceles trapezia and more rarely also into right trapezia. The parallelogram family is divided into rectangles and rhombi, and these two have a common offspring: the square. The kites sometimes have a subclass called right kites, which are defined as kites with two opposite right angles. Sometimes the rhombi are considered to be a subclass of the kites as well as the parallelograms. Quite rarely, the cyclic quadrilateral is added next to the trapezium, with the isosceles trapezium and the rectangle as offspring, see Figure 2 [12. p. 71].

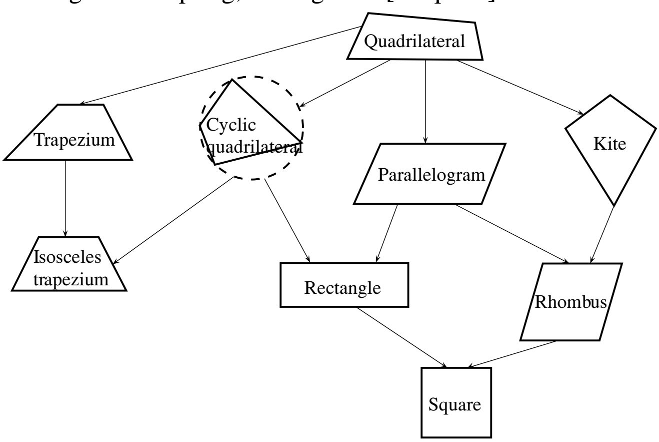

FIGURE 2: A family tree based on the exclusive definition

In a classification based on the *inclusive* definition, the convex quadrilaterals are usually divided into two classes, the trapezia and the kites. The trapezia are divided into isosceles trapezia and parallelograms, with a common subclass: the rectangle. The parallelograms also share a subclass with the kites: the rhombus. And finally there is the square as the offspring to rhombi and rectangles. Quite often an additional class is added at the top next to the trapezium: the cyclic quadrilateral, see Figure 3 [12, p. 69]. It has the isosceles trapezium as a subclass and sometimes also a second called the cyclic kite. The latter is the same as a right kite and has the square as a subclass. In an even more extended classification, the *tangential quadrilateral* (often called a circumscribed quadrilateral; it has an incircle) is added above the kites. Then there is also the intersection between cyclic and tangential quadrilaterals, which are called *bicentric quadrilaterals*. They have the square as an offspring. We note that the classification in Figure 3 is more symmetric than the one in Figure 2, but also that there seems to be something missing in the former. The tree is askew (even if we insert the tangential quadrilateral above the kite) since the trapezium in the middle has a direct connection to the left-hand side but not to the right-hand side.

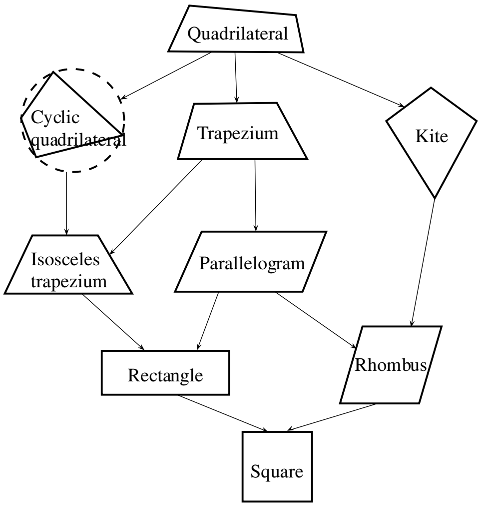

FIGURE 3: A family tree based on the inclusive definition

Roger Wheeler presented in the short note [13] a further extension of the classification of quadrilaterals on the basis of symmetry between sides and angles, which corrected the skewness we just observed. Forty years later the same classification was given by Rudolf Fritsch in [14] (although he, for some strange reason, turned the tree upside down). The convex quadrilateral is divided into the four classes tangential quadrilateral, extangential quadrilateral (see next paragraph), trapezium, and cyclic quadrilateral. Their offspring are in pairs (see Figure 4) the kite, the parallelogram, and the isosceles trapezium, which in turn are the parents in pairs to the rhombus and the rectangle. Finally there is the square at the bottom.

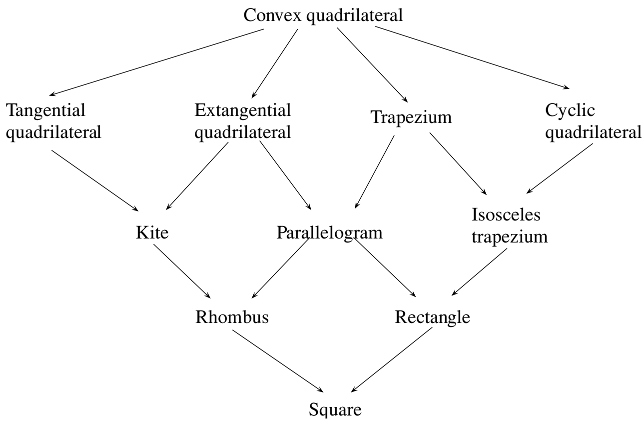

FIGURE 4: Wheeler and Fritsch's classification of convex quadrilaterals

The quadrilateral we have labelled an *extangential quadrilateral* (this name is from [15]) is not so well-known. It was called an ecircular quadrilateral by Wheeler and Fritsch used the German name (Ankreisviereck). We prefer the name extangential quadrilateral due to its close relationship with the tangential quadrilateral. In Fritsch's paper he starts by realising from the side-angle symmetry that these quadrilaterals must have the property that the sum of two adjacent sides is equal to the sum of the other two sides: $a + d = b + c$ when the consecutive sides are labelled a, b, c, d. After an investigation he concludes that this class of quadrilaterals is one with an excircle (Ankreis in German). The equation $a + d = b + c$ is a necessary condition for an excircle outside one of the vertices B or D in a convex quadrilateral $ABCD$, where $a = AB$, $b = BC$, $c = CD$ and $d = DA$. If we consider parallelograms to be extangential quadrilaterals with excircles of infinite radius, then it is also a sufficient condition. There is the similar characterisation $a + b = c + d$ for an excircle outside one of the vertices A or C: see Figure 5. Both criteria can be merged into one as $|a - c| = |b - d|$, which is the complete necessary and sufficient condition for a circle tangent to the *extensions* of all four sides of a convex quadrilateral. If you want to know more about extangential quadrilaterals, we recommend you to check out the freely available online paper [5], where you will find various interesting and beautiful equations that are characteristic for this class.

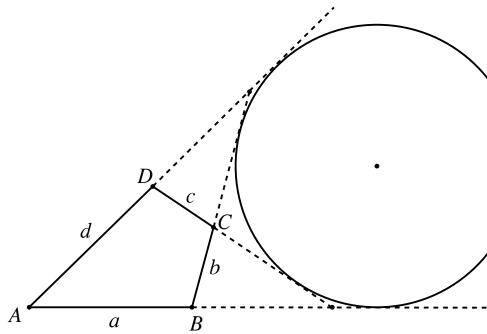

FIGURE 5: An extangential quadrilateral with its excircle outside of C

For some unknown reason neither Wheeler nor Fritsch included the bicentric quadrilateral in their classification. We resolve this inadvertence in our quadrilateral family tree, which will incorporate all of the quadrilaterals in Figure 4 and also several other important classes.

The next classification belongs to another German mathematician, Günter Graumann. He actually presents two different classifications of quadrilaterals in [16]. The first tree consists of four levels, where he has placed trapezium, string-quadrilateral (cyclic quadrilateral), tangent-quadrilateral, sloping-kite and tilted-kite on the top (the meaning of the latter two will soon be explained). On the second level there are right-angled trapezium, symmetric trapezium, parallelogram, kite, and right-angled tilted-kite. Level three includes rectangle, rhombus, and right-angled kite; and finally the square is at the bottom. This diagram shows what Graumann considers to be 'all important convex quadrilaterals': see Figure 6. The sloping-kite is defined to be a quadrilateral where at least one diagonal bisects the other diagonal. However, in his second classification this is called a sliding-kite, so there is some confusion here. Also, it is not crystal clear what Graumann means by tilted-kite, but from what we can understand it's a quadrilateral with two opposite equal angles. These generalised kite-classes are very rarely found in the literature on geometry, so they definitely deserve more attention and exploration.

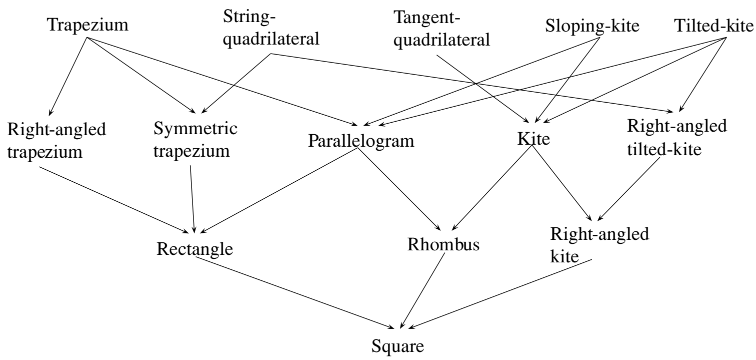

FIGURE 6: Graumann's first house of quadrilaterals

In his second 'house of quadrilaterals', Graumann makes a classification based on properties of the diagonals. We will comment on a few of the classes that are not obvious or well-known. At the top he puts orthogonal quadrilateral (with perpendicular diagonals, which we will call *orthodiagonal quadrilaterals*, as we previously have in [4]), sliding-kite (one diagonal bisects the other), and an unnamed quadrilateral with diagonals of equal length (called an *equidiagonal quadrilateral* in [7]). The orthodiagonal and equidiagonal quadrilaterals have an interesting offspring: a quadrilateral with both perpendicular and equal diagonals. It is not named by Graumann, but in [7, p. 137] we called it a *midsquare quadrilateral* since the midpoints of the sides are the vertices of a square. These quadrilaterals have previously been called skewsquares and pseudosquares according to [17]. They have a special case when one diagonal bisects the other diagonal. Since that quadrilateral is a type of kite (also not named by Graumann), we propose to call it a *midsquare kite*. It is placed between the midsquare quadrilateral and the square, see Figure 7. Note that the names with an asterisk in this figure were not used by Graumann.

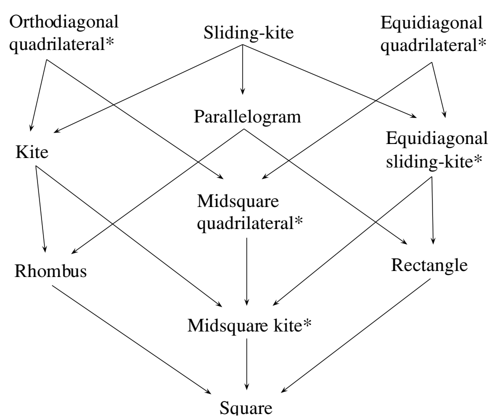

FIGURE 7: Graumann's house of quadrilaterals concerning diagonals

The last previous contribution to the classification of quadrilaterals we study is due to the South African mathematician Michael de Villiers. In [18] and [19, p. 154] he presented an extended classification containing 26 different classes of convex quadrilaterals, many of which have neither been named nor studied anywhere else as far as we know. A reduced version of this tree was presented by him at [20], which is shown in Figure 8. There the convex quadrilateral is divided into six classes: circumscribed quadrilateral, perpendicular quadrilateral (it has perpendicular diagonals), bisecting quadrilateral (at least one diagonal bisects the other diagonal), trapezium, diagonal quadrilateral (it has equal diagonals), and cyclic quadrilateral. The first three have the kite as offspring, the last three have the isosceles trapezium as offspring, and the middle two have the parallelogram as offspring. The kite is subdivided into three classes: rhombus, triangular kite (a kite with at least three equal angles), and right kite, whereas the isosceles trapezium is subdivided into isosceles circumscribed trapezium (trapezium with both an incircle and a circumcircle: a bicentric trapezium), trilateral trapezium (isosceles trapezium with at least three equal sides), and the rectangle. At the bottom we find the square. We shall give a few additional comments on this classification at the end of this paper.

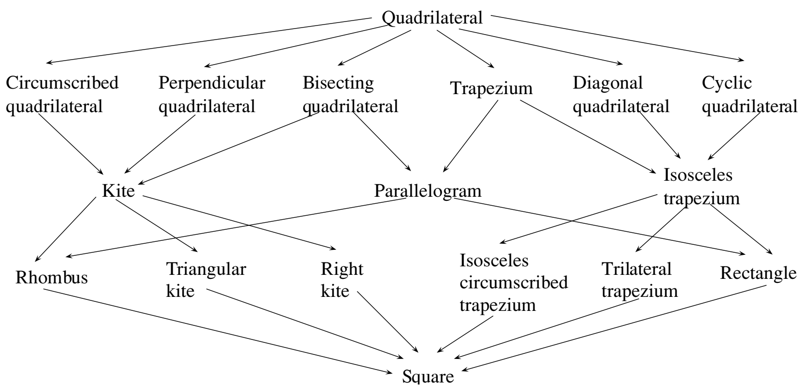

FIGURE 8: De Villiers' classification of convex quadrilaterals

3. *The battle of the trapezium*

We have on the first page of [6] compared the exclusive and the inclusive definition of a trapezium and argued why the latter is preferable from a mathematical point of view. According to the inclusive definition, *a trapezium is a quadrilateral with* ***at least*** *one pair of opposite parallel sides.* Although there are no wrong definitions as long as they don't contradict other definitions or previous theorems, there certainly are better and worse ones. Traditions can be deeply rooted, which is probably a major reason why many contemporary textbooks still define trapezia to be quadrilaterals with exactly one pair of parallel sides and thus exclude the parallelograms from being a subclass. As long as such bad definitions are taught in schools, students will continue to be misled by teachers and textbook authors that disregard the contradiction of using the exclusive definition on trapezia when it is the conventional method to use inclusive definitions in other parts of mathematics, for instance in the definitions of the number system.

It's a sad result that of 86 American high school geometry textbooks studied in [12], 76 used the exclusive definition for the trapezium, only 8 used the inclusive definition, and 2 gave no definition. What is even more hair-raising is the inconsistency when textbooks using the exclusive definition on trapezia have no problem using the inclusive definition for the other quadrilaterals, so for instance the rhombi are a subclass of the parallelograms. If you prefer the exclusive definition, we strongly encourage you to check out Richard Rusczyk's video presentation (especially the last five minutes) at [21]. It will surely give you an Eureka moment.

Why the exception for trapezia? Michael Keyton, at the Illinois Mathematics and Science Academy, advocates the inclusive definition in [22] and proposes the following explanations for the prevalent use of the exclusive definition:

- Many textbook authors do not wish to go against standard terminology;
- Most of the authors have not thought about the inconsistency in terminology;
- These authors are not actively engaged in discovering and proving theorems in geometry;
- There are no converses for the trapezium covered in high school geometry.

Let us consider the fourth point in more detail. In [6] we proved many necessary and sufficient conditions for a convex quadrilateral to be a trapezium. As an example, one of these states that 'a convex quadrilateral is a trapezium if, and only if, one bimedian divides it into two quadrilaterals with equal areas,' where a bimedian is a line segment connecting the midpoints on two opposite sides. The characterisation is described very compactly in this way using inclusive definitions, but when using exclusive definitions, it's no longer a characterisation of trapezia. Furthermore, it will have to be split into two theorems, where the converse must be restated in the form: a convex quadrilateral where one bimedian divides it into two quadrilaterals with equal areas is either a trapezium or a parallelogram. Do we really want to have it this way? The answer is obviously *no*!

We have found one more possible explanation for the use of the exclusive definition when searching the Internet on this debate. There are some interesting discussions to be found online promoting the inclusive definition, but difficult to find comments from someone who is defending the exclusive definition. The proposed explanation ([23], comment on 16 March, 2011) was that the exclusive definition is used so that the statement 'in an isosceles trapezium, the base angles are equal' is true. A problem with the inclusive definition of a trapezium can be that if we define an isosceles trapezium to be a trapezium with two opposite equal sides, then the general parallelogram would also be an isosceles trapezium. Most mathematicians would probably agree that this is undesirable, since a general parallelogram lacks several of the properties of an isosceles trapezium such as equal diagonals, an axis of symmetry, and a circumcircle.

There is however a very easy solution for this situation that is used in some textbooks. Simply define an isosceles trapezium to be a trapezium with two equal base angles. Then the fact that the lateral sides are equal will be a theorem to prove instead of the property that the base angles are equal. We think an even better option is to make the following entirely different definition (found in [24, p. 30] and [25, p. 720]): *an isosceles trapezium is a quadrilateral with two distinct pairs of adjacent equal angles*,* see Figure 9. Thus it can be defined without even using the word trapezium and the issue of having the parallelograms as a subset of isosceles trapezia is over and done with!

FIGURE 9: The preferred definition of an isosceles trapezium

That such a quadrilateral is a trapezium is a direct consequence of this definition via the angle sum of a quadrilateral and basic properties of parallel lines. Other well-known properties like two opposite equal sides and equal diagonals are then easy to prove. A great advantage is that this latter definition makes the side-angle duality between the kite and the isosceles trapezium very clear. The one small compromise either of these two alternative definitions entail is that the isosceles trapezium is defined in a different way from the isosceles triangle. But that is a small price to pay when the gain is a more consistent, more symmetric, and more beautiful family tree for convex quadrilaterals. As the British mathematician G. H. Hardy put it in his book [26]: 'Beauty is the first test: there is no permanent place in the world for ugly mathematics'.

We conclude this discussion by citing Mr Chase's final summation from his passionate blog [23] where he explains why the inclusive definition of a trapezium is better than the exclusive:

- All other quadrilaterals are defined in the inclusive way;
- The area formula for a trapezium still works, even if the legs are parallel;

* Distinct means that the equal angles in the quadrilateral *ABCD* are the pairs *A*, *B* and *C*, *D* or *A*, *D* and *B*, *C*. This does not exclude the possibility of three (and thus four) equal angles.

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- The trapezium approximation method in calculus doesn't fail when one of the trapezia is actually a rectangle;
- When proving that a quadrilateral is a trapezium, one can stop after proving just two sides are parallel.

In a way his last point is perhaps the most important. As he concludes: 'with the exclusive definition, in order to prove that a quadrilateral is a trapezium, you would have to prove two sides are parallel *and* the other two sides are *not* parallel.' This is a serious disadvantage!

4. *Definitions of eighteen classes of quadrilaterals*

Since a few of the quadrilaterals that we include in our classification are not so well-known, we list one defining condition for each of the 18 classes in Table 1 before we present our tree. The abbreviation Q stands for quadrilateral in the table.

Quadrilateral	*A defining condition*
Orthodiagonal quadrilateral	Q with perpendicular diagonals
Tangential quadrilateral	Q with an incircle
Extangential quadrilateral	Q with an excircle
Trapezium	Q with at least one pair of parallel sides
Cyclic quadrilateral	Q with a circumcircle
Equidiagonal quadrilateral	Q with equal diagonals
Tangential trapezium	Trapezium with an incircle
Bicentric quadrilateral	Q with an incircle and a circumcircle
Exbicentric quadrilateral	Q with an excircle and a circumcircle
Kite	Q with two pairs of adjacent equal sides
Parallelogram	Q with opposite parallel sides
Isosceles trapezium	Q with two pairs of adjacent equal angles
Right kite	Kite with two opposite right angles
Bicentric trapezium	Isosceles trapezium with an incircle
Rhombus	Q with four equal sides
Midsquare quadrilateral	Q with equal and perpendicular diagonals
Rectangle	Q with four equal angles
Square	Q that is both a rhombus and a rectangle

TABLE 1: Definitions of eighteen classes of quadrilaterals

For most classes of quadrilaterals there are several known necessary and sufficient conditions. It is important to realise that the defining condition and all such known characterisations for each class are equivalent when using *inclusive* definitions, so any one of them could equally well be used as the defining condition. This is the main reason why there are various different definitions in use for several of the classes of quadrilaterals (as discussed in [12]). Note in particular that exclusive and inclusive definitions are not equivalent. If we want to teach students the beauty of geometry, we should all be engaged in the banning of exclusive definitions!

In our hierarchical classification, we only consider *plane convex quadrilaterals*. By a plane quadrilateral we mean the figure formed by connecting four points in a plane with straight line segments, of which no three are collinear and no two coincide. It is convex if the two diagonals both lie inside the quadrilateral. Some of the classes also have concave versions (one exterior diagonal) and crossed ones (self-intersecting, with two exterior diagonals), but these are not considered here.

5. *Our hierarchical classification of quadrilaterals*

When constructing our family tree for convex quadrilaterals, we were greatly inspired by the classifications due to Fritsch [14] and de Villiers [20]. We have tried to combine their best ideas and also to include a few missing classes that we consider to be important. Our selection of quadrilaterals is based on two principles. For one thing, the tree shall have a *mirror symmetry* in the vertical midline regarding sides and angles, or perpendicular and equal diagonals. Secondly, the quadrilaterals included shall be interesting enough to have been studied in geometry books or papers, so that at least some of their properties are known. This made us exclude a few quadrilaterals (for instance the right trapezium) that did not meet both of these criteria.

On the other hand, we have included the bicentric quadrilateral and its relative, the *exbicentric quadrilateral*, as well as the *tangential trapezium* and the *bicentric trapezium* (see Table 1). These are rarely found in previous classifications of quadrilaterals. They were included since we think the tangential, extangential and cyclic classes are interesting and deserve more attention in textbooks and problem solving than they have previously received.

In our quadrilateral family tree, the convex quadrilateral is divided into six classes (note that only five of them are the same as in de Villiers' classification): orthodiagonal quadrilateral, tangential quadrilateral, extangential quadrilateral, trapezium, cyclic quadrilateral, and equidiagonal quadrilateral. Figure 10 indicates their offspring. On the second level below the top we find the tangential trapezium, the bicentric quadrilateral, and the exbicentric quadrilateral. On the third level there are three well-known quadrilaterals: kite, parallelogram, and isosceles trapezium. Level four includes five quadrilaterals: rhombus, right kite, midsquare quadrilateral, bicentric trapezium, and rectangle. These five classes have the square as their common special case, which completes our hierarchical classification.

We are confident that when considering their definitions from Table 1 or in same cases a well-known characterisation, it is apparent that many of the quadrilaterals in our classification exhibit a symmetry regarding sides and angles between the left and right-hand side. For example, the sums of opposite sides are equal in a tangential quadrilateral and the sums of opposite angles are equal in a cyclic quadrilateral. The sums of two adjacent sides are equal in an extangential quadrilateral and the sums of two adjacent angles are equal in a trapezium. Two pairs of adjacent sides are equal in a kite and two pairs of adjacent angles are equal in an isosceles trapezium. When it comes to the diagonals, many of the quadrilaterals on the left-hand side have perpendicular diagonals and the ones on the corresponding positions on the right-hand side have equal diagonals. Another symmetry is about the existence of an incircle (to the left) or a circumcircle (to the right). On the lower levels we have not drawn all of the incircles and circumcircles,* but the symmetry is present all the way down to the square.

It is also interesting to note that the right kite and the bicentric trapezium have the properties of maximal and minimal area respectively among all bicentric quadrilaterals with a given incircle and circumcircle, and that the area of these quadrilaterals can be expressed in terms of the inradius and the circumradius alone (see [27, 28]).

6. *About a few classes that were excluded*

At the bottom of our quadrilateral family tree, it would be possible to include two quadrilaterals between the midsquare quadrilateral and the square. These are neither well-known nor previously named, but suitable names are *midsquare kite* and *midsquare trapezium*. The first, which is a kite with equal diagonals (and therefore called an equidiagonal kite by us in [7, p. 137]), was mentioned at the end of Section 2. The second is an isosceles trapezium with perpendicular diagonals, i.e. an orthodiagonal isosceles trapezium. They are both special cases of a midsquare quadrilateral, or, considered from the other direction, generalisations of a square.

Two other generalisations of a square are included in de Villiers' classification at [20] (our Figure 8). They are the *trilateral trapezium* and *triangular kite*. If the triangular kite, the midsquare kite, the midsquare trapezium, and the trilateral trapezium were to be included in our tree (Figure 10), they could all be placed in this order on a fifth level below the top (the convex quadrilateral is on level zero), putting the square on a new level six.

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* Kites and rhombi have an incircle, isosceles trapezia and rectangles have a circumcircle, and right kites and squares have both of these circles.

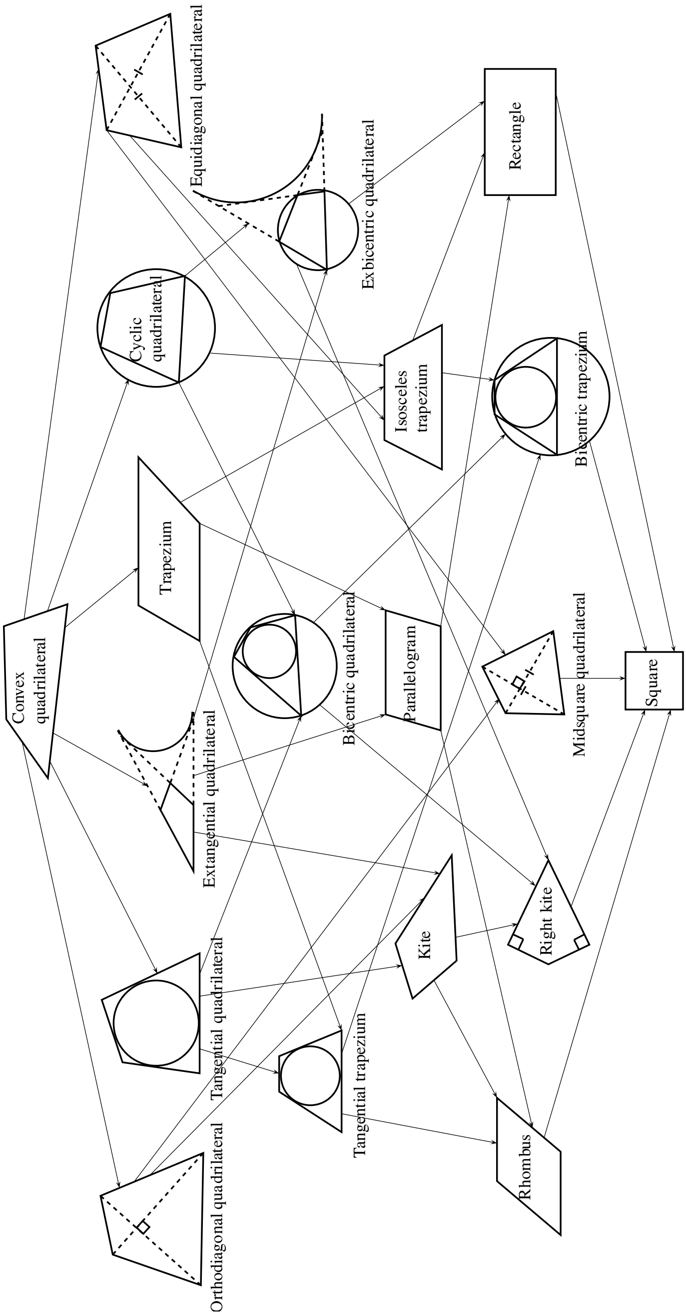

FIGURE 10: Our quadrilateral family tree including the eighteen most important classes of convex quadrilaterals

Graumann defined in [16] a couple of classes of generalised kites which have been little studied. The sloping-kite and tilted-kite are called bisecting quadrilateral and angle quadrilateral respectively by de Villiers in his extended classification (see [18] or [19, p. 154]) containing 26 different classes of convex quadrilaterals. This is an interesting extension starting from a side-angle duality. There is nevertheless one peculiarity in de Villiers' classification. He puts the bisecting quadrilateral (a quadrilateral where at least one diagonal bisects the other diagonal) as the symmetric partner to the trapezium. This cannot be right. It is evident that the correct dual partner from a side-angle perspective is an extangential quadrilateral as in the Wheeler/Fritsch classification (see Figure 4). A drawback in de Villiers' extended classification is that properties of several of his included quadrilaterals have neither been studied in his book [19] nor anywhere else that we know of, and this is the main reason why we did not included them in our tree.

It is likely that the most well-known quadrilateral that was excluded from the family tree in Figure 10 is the *right trapezium*. If you think it is an important quadrilateral that should be included, then it can be placed to the right of the isosceles trapezium and above the rectangle, which is an offspring (and with the trapezium as a parent). But then the mirror symmetry will be broken since it has no named dual partner.

7. *There is nothing new under the sun*

Sometimes strange coincidences happen. The day after we finished a revised version of this paper, we received from a German friend an e-mail with a copy of a fifty-year-old German paper [29] about extangential quadrilaterals (which was the topic of our correspondence). Imagine our surprise when we found in that paper almost the same classification of quadrilaterals that we had drawn in Figure 10. Not so surprising you might think, but the thing is that this friend had no idea we were writing the present paper on quadrilateral classification. We drew Figure 10 a year and a half before, and then that old paper arrived out of the blue just as punctually as a German train. The truth of the old saying in this section's heading is evident.

There was a small difference between the two classifications. Wolfgang Zirkel (yes, it is German for circle, and he wrote about circles associated with quadrilaterals) left out the orthodiagonal quadrilateral, the equidiagonal quadrilateral, and the midsquare quadrilateral. If we disregard them and then put, as Zirkel did, the six quadrilaterals in the two middle levels of our tree in Figure 10 on a new central level, then that classification has the interesting level formation 1, 4, 6, 4, 1 concerning the number of quadrilaterals on each level. This is a sequence from a famous triangle known in India at least a millennia before Pascal rediscovered it (there is nothing new . . .).

What is the connection between Pascal's triangle and Zirkel's quadrilateral classification? Zirkel considered the four angle and side characterisations $A + C = B + D$, $a + c = b + d$, $|A - C| = |B - D|$, and $|a - c| = |b - d|$ of a convex quadrilateral $ABCD$ with consecutive sides a, b, c, d (however he neglected absolute values in the latter two equations). These characterisations correspond to cyclic quadrilaterals, tangential quadrilaterals, trapezia, and extangential quadrilaterals respectively. Then it is simply a combinatorial problem of choosing how many quadrilaterals that satisfy 0, 1, 2, 3, 4 of these equations. Hence we get the fourth row binomial coefficients 1, 4, 6, 4, 1.

It could be interesting to note that there is an altogether different way of presenting the 16 quadrilaterals in Zirkel's classification. All but one of these 16 are descendants of either the cyclic quadrilateral, the tangential quadrilateral, the trapezium or the extangential quadrilateral. Table 2 shows this alternative interpretation, which is not from [29]. In all fairness, it is not our original idea either. The inspiration came from Jürgen Köller's German website [30] (he attributes it to Peter Jirjahlke), where the class called bisecting quadrilateral by de Villiers was present instead of the extangential quadrilateral, and its cyclic version instead of the exbicentric quadrilateral compared to Table 2. (The quadrilaterals in the two left columns were then considered to be different types of kites in a more general way than is customary.)

	Extangential quadrilaterals				
Trapezia	Parallelogram	Rhombus	Tangential trapezium	Trapezium	
	Rectangle	Square	Bicentric trapezium	Isosceles trapezium	Cyclic quads
	Exbicentric quadrilateral	Right kite	Bicentric quadrilateral	Cyclic quadrilateral	
	Extangential quadrilateral	Kite	Tangential quadrilateral	Convex quadrilateral	
		Tangential quadrilaterals			

TABLE 2: A different interpretation of Zirkel's quadrilateral classification

Another thing worth mentioning is that the paper by Zirkel also contained on its first page the same classification we have attributed to Wheeler and Fritsch. Like his compatriot many years later, Zirkel also drew the tree upside down.

8. *Concluding remarks*

Apart from presenting our hierarchical classification of convex quadrilaterals, this paper has aimed to serve other purposes. One was to revisit the battlefield of the trapezium and emphasise a rarely used definition of an isosceles trapezium. Hopefully we might convince some future textbook authors to choose the inclusive definition of trapezia instead of the improper exclusive definition. But more importantly, we wanted to promote the interest in quadrilateral geometry, for teachers of geometry at all levels of education as well as for researchers at universities and hobby geometers. Properties and characterisations of quadrilaterals are fascinating subjects in the beautiful realm of geometry, and there still exists several unexplored parts of their hierarchy as briefly discussed in this paper. It is truly exciting to be a contributor in this field and to help expand the knowledge about Euclidean geometry accumulated over the past two and a half millennia.

References

1. M. Josefsson, Characterizations of bicentric quadrilaterals, *Forum Geom.* **10** (2010) pp. 165-173.
2. M. Josefsson, More characterizations of tangential quadrilaterals, *Forum Geom.* **11** (2011) pp. 65-82.
3. M. Josefsson, When is a tangential quadrilateral a kite?, *Forum Geom.* **11** (2011) pp. 165-174.
4. M. Josefsson, Characterizations of orthodiagonal quadrilaterals, *Forum Geom.* **12** (2012) pp. 13-25.
5. M. Josefsson, Similar metric characterizations of tangential and extangential quadrilaterals, *Forum Geom.* **12** (2012) pp. 63-77.
6. M. Josefsson, Characterizations of trapezoids, *Forum Geom.* **13** (2013) pp. 23-35.
7. M. Josefsson, Properties of equidiagonal quadrilaterals, *Forum Geom.* **14** (2014) pp. 129-144.
8. M. Josefsson, A few inequalities in quadrilaterals, *International Journal of Geometry* **4** no. 1 (2015), pp. 11-15.
9. D. E. Joyce, *Euclid's Elements*, Book I (1996), http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html
10. A. Athanasopoulou, *An inquiry approach to the study of quadrilaterals using Geometer's Sketchpad: a study with pre-service and in-service teachers*, Proquest, Umi Dissertation Publishing (2011).
11. T. K. Puttaswamy, *Mathematical achievements of pre-modern Indian mathematicians*, Elsevier Science Publishing (2012).
12. Z. Usiskin and J. Griffin, *The classification of quadrilaterals. A study of definition*, Information Age Publishing (2008).
13. R. F. Wheeler, Quadrilaterals, *Math. Gaz.* **42** (December 1958) pp. 275-276.
14. R. Fritsch, Bemerkungen zur Viereckslehre, *Schriften der Sudetendeutschen Akademie der Wissenschaften und Künste*, **19** (1998) pp. 69-94, also available at http://www.mathematik.uni-muenchen.de/~fritsch/viereck.pdf

15. M. Radic, Z. Kaliman and V. Kadum, A condition that a tangential quadrilateral is also a chordal one, *Math. Commun.* **12** (2007) pp. 33-52.
16. G. Graumann, Investigating and ordering quadrilaterals and their analogies in space − problem fields with various aspects, *ZDM*, **37** 3 (2005), pp. 190-198, also available at http://subs.emis.de/journals/ZDM/zdm053a8.pdf
17. A. T. Olson, Exploring skewsquares, *Mathematics Teacher* **69** (November 1976) pp. 570-573.
18. M. de Villiers, *An extended classification of quadrilaterals* (1996), available at http://mysite.mweb.co.za/residents/profmd/quadclassify.pdf
19. M. de Villiers, *Some adventures in Euclidean geometry*, Dynamic Mathematics Learning (2009).
20. M. de Villiers, *Hierarchical quadrilateral tree* (2011), available at http://frink.machighway.com/~dynamicm/quad-tree-web.html
21. R. Rusczyk, *Art of problem solving: classifying quadrilaterals* (2012), available at https://www.youtube.com/watch?v=xoXLb0WRBMA
22. M. Keyton, *Trapezoids Talk* (2008), available at http://www.fresnarus.com/keyton/Mathematical%20Talks%20and%20Papers/Trapezoids%20Talk.pdf
23. Mr. Chase, *Random Walks. Mr. Chase blogs about math* (2011), available at http://mrchasemath.wordpress.com/2011/02/03/why-i-hate-the-definition-of-trapezoids/
24. T. V. Craine and R. N. Rubenstein, A quadrilateral hierarchy to facilitate learning in geometry, *Mathematics Teacher* **86** (January 1993) pp. 30-36.
25. M. de Villiers, Using dynamic geometry to expand mathematics teachers' understanding of proof, *Int. J. Math. Educ. Sci. Technol.* **35** 5 (2004), pp. 703-724.
26. G. H. Hardy, *A mathematician's apology*, Cambridge University Press (1940).
27. M. Josefsson, Maximal area of a bicentric quadrilateral, *Forum Geom.* **12** (2012) pp. 237-241.
28. M. Josefsson, Minimal area of a bicentric quadrilateral, *Math. Gaz.* **99** (July 2015) pp. 237-242.
29. W. Zirkel, Das Ankreisviereck, *Praxis der Mathematik* **5** (1963) pp. 129-132.
30. J. Köller, *Hierarchie der Vierecke* (2007), available at http://www.mathematische-basteleien.de/viereck.htm

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