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J. Karakonstantis & T. Patronis

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Relational understanding and paths of reasoning through a Boolean lattice classification of quadrilaterals

J. Karakonstantis* and T. Patronis

Department of Mathematics, University of Patras, Patras, Greece

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In this article we study a specific lattice classification of converse quadrilaterals, based on the relations between diagonals. This lattice contains $16 = 2^4$ elements, which form a hypercube, and therefore it is a Boolean lattice. 'Complementary' species of quadrilaterals thus appear and may be related in the lattice diagram. We also introduce and study the idea of *paths of reasoning*, which may connect distinct species of quadrilaterals in various ways, thus leading to a relational understanding of these concepts.

Keywords: concept lattice; Boolean; hypercube; complementary elements; relations of diagonals; relational understanding

1. Introduction and background

Since the time of Humboldtian reform in education, 'organic' or 'systematic' thinking has been emphasized as necessary for understanding 'the spirit of mathematics' [1, pp. 60–61].

Aiming at an 'organic' understanding of quadrilaterals, as a subject of geometry teaching, several authors have classified them by using *Venn diagrams* or *lattice diagrams*, according to different criteria. Perhaps, the lattice classification diagrams are more significant, because they may indicate various paths of reasoning, thus functioning as representations or models of reasoning and understanding. It is this direction that we wish to explore further in this article, by studying the possibilities offered by a specific lattice classification introduced here. This classification is based on a hierarchy of relations holding between diagonals of convex quadrilaterals and leads to the Boolean lattice of $2^4 = 16$ elements, which can also be represented as a hypercube (see Figures 1 and 4 in Sections 2 and 4, respectively). There are, of course, other useful classifications, based on relations between angles, sides or medians. The one presented here will be studied as an example illustrating *relational understanding* of quadrilaterals and indicating some interesting *paths of reasoning*.

1.1. *Relational understanding in geometry*

In his well-known paper about *relational* and *instrumental understanding*, Skemp [2] used an example from school geometry, namely the statement that the circumference

*Corresponding author. Email: ikarako@sch.gr

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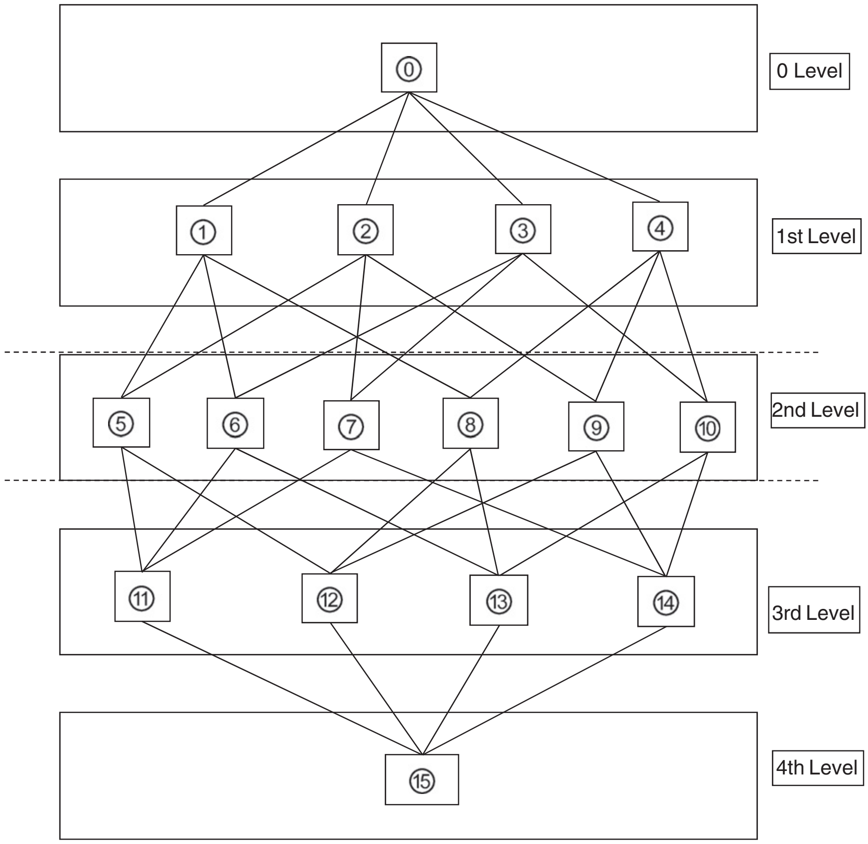


Figure 1. Hierarchical classification of quadrilaterals represented by the Boolean lattice of 16 elements.

of a circle of diameter d is equal to πd. He first sketched an instrumental explanation and then a plan of relational understanding.

Relations in Greek Geometry (since Euclid and even before) often appear in the form of *proportions*, which also shows their independence from the unit of measurement. In the case of Skemp's example, the relation 'Circumference $= \pi d$' could be rewritten in proportional form as follows:

$$\frac{\text{Circumference (1)}}{\text{Diameter (1)}} = \frac{\text{Circumference (2)}}{\text{Diameter (2)}}$$

This is a relation holding for any two circles (1) and (2) and leading to the geometrical definition of the number π.

On the other hand, we have *logical interconnections between geometrical concepts* (such as between different species of triangles or quadrilaterals). These logical interconnections lead to relational understanding at an advanced level of thinking. We may refer to equality, geometric proportion or other relations between geometrical magnitudes as 'first-order geometric relations', while we can speak of logical relations between classes of geometrical figures as 'second-order geometric relations'. Understanding of first-order geometric relations correspond to lower van Hiele levels and naturally precede the understanding of second-order geometric relations; otherwise it seems very difficult to conceive a square as a rhombus, a rectangle as a parallelogram, etc., and thus to arrive to a hierarchical classification of quadrilaterals.

1.2. *Concept lattices and duality*

According to Sierpinska [3, Chapter 3], a process of understanding can be represented as 'a lattice of the acts of understanding linked by reasonings'. Thus if **A** and **B** are acts of understanding, then in this lattice we have $\mathbf{A} \leq \mathbf{B}$ if there is a reasoning $\mathbf{R_{AB}}$ that leads from **A** to **B**. This idea relates *understanding* to *concept lattices* as introduced and studied by Wille [4]. In a concept lattice we have an ordering relation between concepts; and concepts, being the 'products' of acts of understanding, can be identified to these acts in a lattice representation, provided that the reasoning $\mathbf{R_{AB}}$, leading from an act of understanding **A** to an act of understanding **B** also leads from the concept 'produced' by **A** to an understanding of the concept 'produced' by **B**.

By representing the acts of understanding (or concepts) as elements of a lattice, we can relate understanding with mathematically complex structures and their specific ordering relations. What seems to be a little more intriguing here, that is a lattice diagram induces a kind of *symmetry* or *duality* between intension and extension of scientific concepts. Indeed, if we look at the lattice diagram of a hierarchy of concepts, the point corresponding to the least element of the lattice can be considered as representing a concept of minimum extension. But minimum *extension* implies a maximum complexity at the *intentional* level, which means that the same concept is the 'richest' one in properties of all the hierarchy.

1.3. *Focus of this article*

The idea of lattice ordering of mathematical concepts has appeared several times in the teaching of quadrilaterals, and its didactical role has been studied independently by De Villiers [5] and Patronis and Spanos [6]. De Villiers' study focused on previous empirical research [5,7], in which the learners were able to follow the logic of hierarchical classification of quadrilaterals, but they were unwilling to accept it. Also Gagatsis et al. [8] reinterpreted similar learners' difficulties by introducing a *semantic distance* between concepts as a psychological and cultural factor in the understanding of concept hierarchies. In this article we adopt a more 'dynamic' point of view, by studying various *paths* connecting concepts in a particular lattice classification of quadrilaterals based on relations between diagonals. We consider these paths as 'paths of reasoning' linking geometrical concepts. Our classification leads to a Boolean lattice; thus we also introduce and study the didactical role of 'complementary species' of quadrilaterals, as another contribution to relational understanding.

2. A lattice classification based on relations between diagonals

A relation of *lattice order* $\leq$ can be defined between species of quadrilaterals by using the following relations between diagonals of any given (convex) quadrilateral:

R_1: The relation of equality between diagonals.
R_2: The relation of perpendicularity between diagonals.

R_3: The relation in which one of the diagonals bisects the other.
R_4: The relation of equality of ratios of the sections into which diagonals cut each other.

The lattice order $\leq$ is then defined as follows: We say that 'the concept q_1 is subordinate to the concept q_2 and we write $q_1 \leq q_2$ if and only if, whenever one of the relations R_1, R_2, R_3 and R_4 holds between the diagonals of a q_2-quadrilateral, the same relation also holds between the diagonals of a q_1-quadrilateral.[1]

By using the lattice order $\leq$, as a classification criterion, we classify convex quadrilaterals in the lattice diagram given in Figure 1. The numbered vertices in this diagram stand for the following concepts:

(0) The concept of convex quadrilateral.
(1) The concept of convex quadrilateral with equal diagonals.
(2) The concept of a convex quadrilateral, in which *at least one* diagonal bisects the other.
(3) The concept of *trapezium* (diagonals intersect each other in equal ratios).
(4) The concept of a quadrilateral with perpendicular diagonals.
(5) The concept of a quadrilateral with equal diagonals, in which at least one diagonal bisects the other.
(6) The concept of *isosceles trapezium* (diagonals are equal and intersect each other in equal ratios).
(7) The concept of *parallelogram* (diagonals bisect each other).
(8) The concept of a quadrilateral with equal and perpendicular diagonals.
(9) The concept of a *kite* (diagonals are perpendicular and one of them bisects the other).
(10) The concept of a trapezium with perpendicular diagonals.
(11) The concept of *rectangle* (diagonals are equal and bisect each other).
(12) The concept of a quadrilateral with equal and perpendicular diagonals, one of which bisects the other.
(13) The concept of a trapezium with equal and perpendicular diagonals which cut each other in equal ratios.
(14) The concept of *rhombus* (diagonals are perpendicular, and bisect each other).
(15) The concept of *square* (diagonals are equal, perpendicular and bisect each other).

Concepts of the lattice diagram in Figure 1 are naturally classified at five levels, as follows. The general concept of quadrilateral is classified at the 0-Level. The concepts of quadrilaterals with one specific appearance of the relations R_1, R_2, R_3 and R_4 are classified at the 1st level and so on. According to the lattice order relation $\leq$, a concept of a higher level is related to a concept of a lower level in the diagram only if the lower level concept shares all properties of the higher level concept (and some more).

3. Concept derivation trees and paths of reasoning

Relational understanding is not an information to be 'conveyed' to students once and for all. It is not by the direct teaching of lattice diagrams, but only gradually, by reasoning and inference procedures, that students can conquer logical interconnections between concepts.

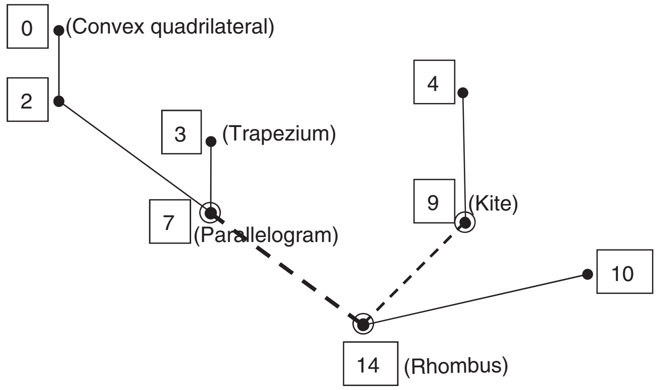


Figure 2. A concept derivation tree of length 8.

On the other hand, the lattice diagram in Figure 1 can serve as a representational framework (or model) for various *paths of reasoning* (see below). Teaching can move either by logical analysis of the (logically) 'more complex' concepts, starting from the bottom, or, from the (logically) 'simpler' concepts, following a synthetic procedure. Within this context, learning is considered as dialectic between the analysis and synthesis of concepts, in accordance with the duality between the intentional and extensional definition of concepts (which we have discussed in Section 1).

A '*concept derivation tree'* in a concept lattice L may be defined, in general, as any subgraph of the diagram of L which is a *tree* in the graph–theoretic sense, (see, e.g. [9]). The length of a concept derivation tree is then defined by the number of concepts figuring in it. In our case, such a concept derivation tree of length 8 (with the concept of rhombus as its 'root') appears in Figure 2.

As a '*path of reasoning'* we consider any graph–theoretic *path* contained in a concept derivation tree. For example, a path of reasoning in Figure 2 is the dotted path

$$(7) \rightarrow (14) \rightarrow (9)$$

This path represents a reasoning which leads from the concept of parallelogram to that of kite *via* the concept of rhombus: a rhombus thus appears as a common specialization of a parallelogram and of a kite.

3.1. *Square–rectangle–isosceles trapezium*

This path, symbolically $(15)\xrightarrow{\urcorner R_2}(11)\xrightarrow{\urcorner R_3}(6)$, gradually reduces the relational structure (and in particular the symmetry) of the square into the relational structure (and symmetry) of an isosceles trapezium. During this procedure, the extension of concepts is increasing while intentionally they are defined by less properties.

Conversely, the *isosceles trapezium–rectangle–square* path enriches the relational structure of the trapezium by leading to the relational structure of the square.

3.2. *Square–rhombus–kite*

The *square–rhombus–kite* path, symbolically $(15) \xrightarrow{\urcorner R_2} (14) \xrightarrow{\urcorner R_3} (9)$, gradually reduces the relational structure of the square into the relational structure of the kite. This

Longer paths

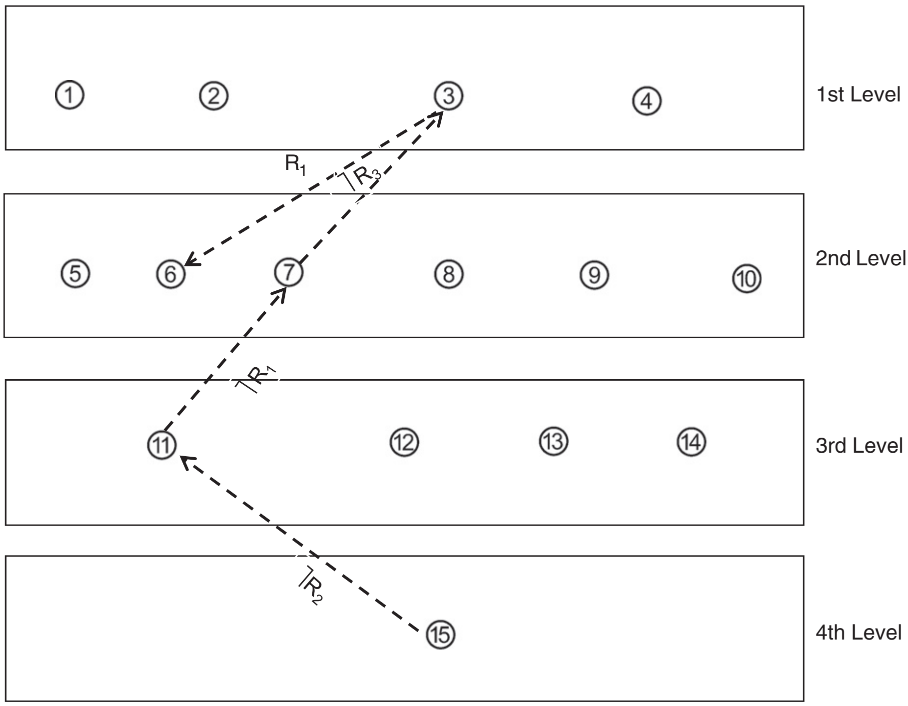


Figure 3. The square-rectangle-parallelogram-trapezium-isosceles trapezium path.

path can be continued until the species (4), which means a convex quadrilateral with perpendicular diagonals. This species can be related to the species of rhombus. The species (9) of kite stands between (14) and (4), which shows that (4) does not 'cover' (14) in the lattice order.

Conversely, the *kite–rhombus–square* path, symbolically (9) $\xrightarrow{\rceil R_3}$ (12) $\xrightarrow{\rceil R_1}$ (15), enriches the relational structure of the kite by first leading to the concept of rhombus and then to the square.

3.3. *Longer paths*

There are now some interesting paths of greater length in our lattice diagram. For example, the *square–rectangle–parallelogram–trapezium–isosceles trapezium* path has total length 5 (Figure 3). Symbolically, (15) $\xrightarrow{\rceil R_2}$ (11) $\xrightarrow{\rceil R_1}$ (7) $\xrightarrow{\rceil R_3}$ (3) $\xrightarrow{R_1}$ (6). During this procedure, the extension of concepts is first increasing and then decreasing (at the last step). This path shows a logical connection between more concepts than the previous one (15) → (11) → (6), and it does this in a quite natural way.

4. 'Complementary' species of quadrilaterals

It is easy to prove that the lattice Q of 16 species of quadrilaterals, described in Section 2, is in fact a Boolean lattice, also represented by the diagram of a discrete hypercube [10].

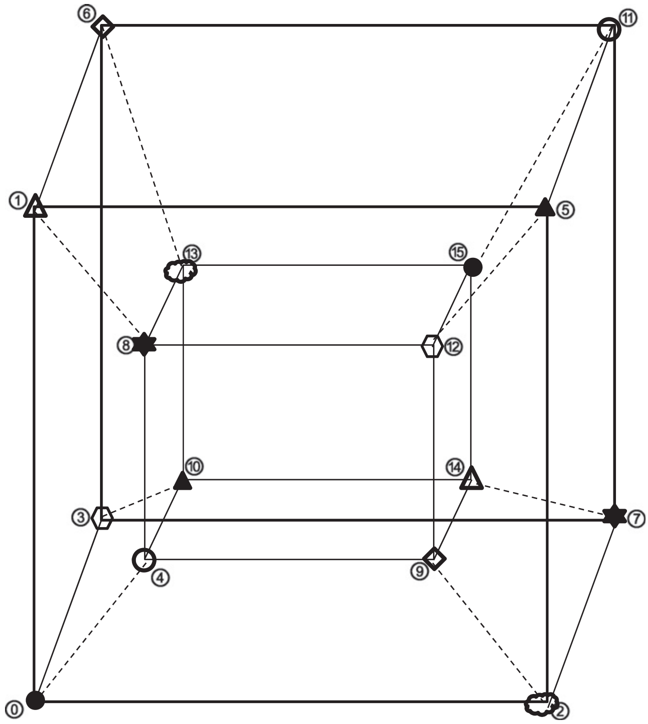


Figure 4. A hypercube representation of the Boolean lattice of 16 elements.

For convenience, we note here all couples of complementary elements in Q:

$\overline{⓪}$ = ⑮	$\overline{④}$ = ⑪
$\overline{①}$ = ⑭	$\overline{⑤}$ = ⑩
$\overline{②}$ = ⑬	$\overline{⑥}$ = ⑨
$\overline{③}$ = ⑫	$\overline{⑦}$ = ⑧

This mathematical fact has an interesting cognitive interpretation. Complementary elements in the lattice Q define 'complementary' species of quadrilaterals, i.e. species of quadrilaterals with completely different properties.

As shown in Figure 4, any two complementary species of quadrilaterals are always situated at two antipodal vertices of the hypercube. In fact, this discrete hypercube diagram can be considered as a *geometrical model* of species of quadrilaterals and their relations, according to Gagatsis and Patronis [11]. This means that geometric properties of the four-dimensional discrete hypercube correspond to the structural properties of the Boolean lattice Q, as well as to their learning and cognition.

As an example, the relation between the complementary elements ① and ⑭ in Q, described by the formulas: ⑭ $\wedge$ ① = ⑮, ⑭ $\vee$ ① = ⓪, can be stated as follows: *a rhombus* *with equal diagonals is a square*. Similar statements corresponding to other complementary elements of Q read as follows:

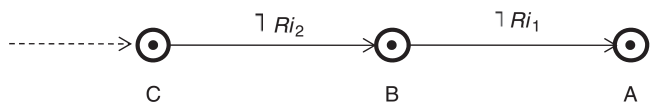
- A rectangle with perpendicular diagonals is a square.
- A kite, which is also an isosceles trapezium, is a square, etc.

5. Concluding remarks

The learning and teaching of convex quadrilaterals which develops in the paths such as $(5) \rightarrow (11) \rightarrow (6)$ and $(15) \rightarrow (14) \rightarrow (9)$ can schematically be represented as follows:

The function of this scheme is the following:

- The concept placed at the position A, (as, e.g. the concept of trapezium) has the role of an initial hypothesis, in a series of 'specializing steps'. By adding successively one more hypothesis at a time, we get a linear procedure, represented by a *chain* in the lattice diagram at a time, in the sense that $A \geq B \geq C \geq \cdots$.
- By reversing the above scheme, it is expected that the students will be involved in a process of reflective abstraction, which is schematically introduced as follows:



There are evident limitations of these linear schemes in the learning and teaching of quadrilaterals. By increasing the number of chain-like paths of reasoning, one does not improve the quality of learning. The quality of learning can only be improved, when an 'organic' connection among the different concepts is developed: it is this type of learning that has been characterized by Skemp as relational or conceptual understanding.

By using various concept derivation trees and paths of reasoning (in particular, between *complementary* species of quadrilaterals such as, e.g. the kite and the isosceles trapezium), it is hoped that a relational cognitive scheme, in the form of acts of understanding, will render the students capable of creating paths by themselves, and 'walk' along them, so that they would be able to evolve from any reason successfully 'starting' concept to any 'ending' one.

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Note

1. In a larger project, which stands beyond the limits of the present article, one can also consider relations among sides and diagonals, as, e.g. the relation R^*: $\delta_1\delta_2 =$ ac+bd, where δ_1, δ_2, are the diagonals of the quadrilateral and a, b, c, d its sides (Ptolemeus theorem), so that the lattice Q could include the quadrilaterals inscribed in a circle.

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