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A Dual to Kosnita's theorem

Michael de Villiers, Mathematics Education,
University of Durban-Westville, Durban 4000, South Africa

In the September 1995 issue of the *Mathematics and Informatics Quarterly*, the following interesting result was mentioned without proof in the column on Forgotten Theorems:

Kosnita's Theorem. The lines joining the vertices *A*, *B*, and *C* of a given triangle *ABC* with the circumcenters of the triangles *BCO*, *CAO,* and *ABO* (*O* is the circumcenter of Δ *ABC*), respectively, are concurrent.

On the basis of an often observed (but not generally true) duality between circumcenters and incenters (eg. see De Villiers, 1996), I immediately conjectured that the following dual to Kosnita's theorem might be true, namely:

Kosnita Dual. The lines joining the vertices *A*, *B*, and *C* of a given triangle *ABC* with the incenters of the triangles *BCO*, *CAO,* and *ABO* (*O* is the incenter of Δ *ABC*), respectively, are concurrent.

Investigation on the dynamic geometry program *Sketchpad* then confirmed that the conjecture was indeed true. I then found that both results could easily be proved by the following useful result for concurrency, a proof of which is given in De Villiers (1996). Interestingly, the point of concurrency for the special case with equilateral triangles on the sides is called the Fermat-Torricelli point.

Fermat-Torricelli Generalization. If triangles *DBA*, *ECB* and *FAC* are constructed on the sides of any triangle *ABC* so that $\angle DAB = \angle CAF$, $\angle DBA = \angle CBE$ and $\angle ECB = \angle ACF$ then *DC*, *EA* and *FB* are concurrent (see Figure 1).

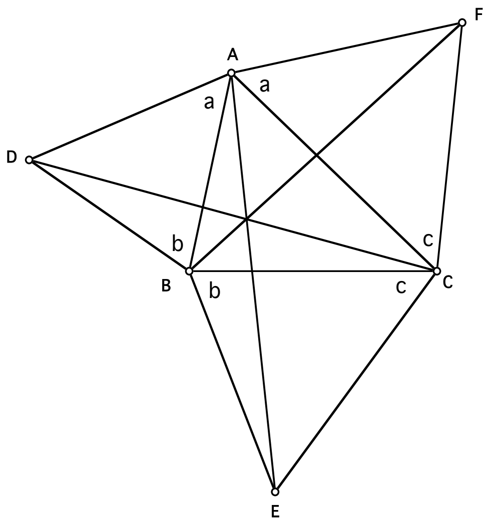

Figure 1

It should further be pointed out that the above result is also true when the triangles are constructed inwardly. As shown in Figure 2 we clearly have for Kosnita's dual that $\angle DAB = \frac{1}{4}\angle A = \angle CAF$, $\angle DBA = \frac{1}{4}\angle B = \angle CBE$ and $\angle ECB = \frac{1}{4}\angle C = \angle ACF$, and from the above result it therefore follows that *DC*, *EA* and *FB* are concurrent.

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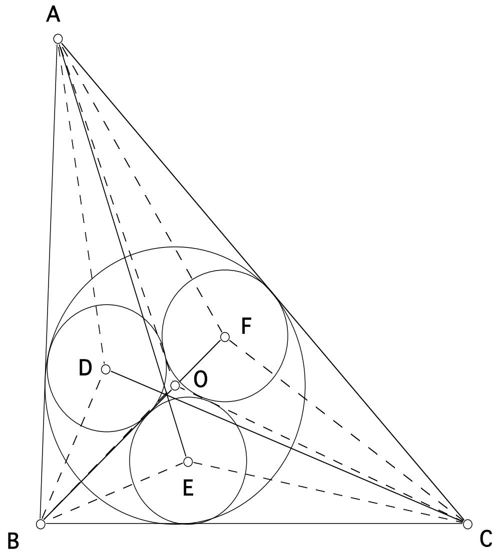

Figure 2

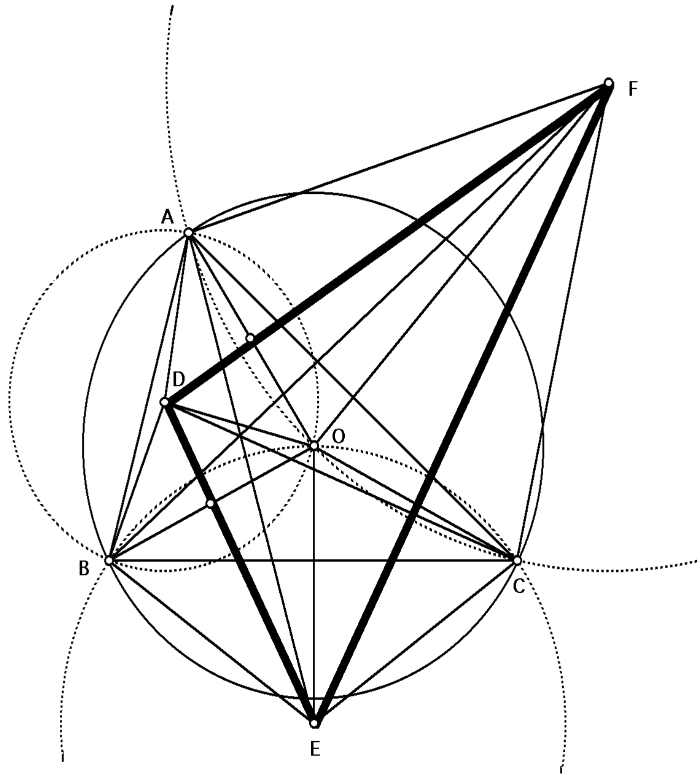

Figure 3

Kosnita's theorem follows a little less directly from the Fermat-Torricelli generalization. In this case, the "base triangle" is triangle *DEF* with *A*, *B* and *C* the outer vertices (see Figure 3). Since *DBOA* is a kite, we have $\angle BDO = \angle ADO$. But *DBEO* and *DOFA* are also kites. Therefore, $\angle BDE = \frac{1}{2}\angle BDO$ and $\angle ADF = \frac{1}{2}\angle ADO$ from which follows that $\angle BDE = \angle ADF$. In a similar fashion can be shown that $\angle BED = \angle CEF$ and $\angle CFE = \angle AFD$. From the F-T generalization, it therefore follows that *DC*, *EA* and *FB* are concurrent.

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Reference

De Villiers, M. (1996). *Some Adventures in Euclidean Geometry*. Durban: University of Durban-Westville.

[**After Note**: I later realized that the dual result for Kosnita's theorem is also valid if the excentres of a triangle ABC are constructed, then the 3 lines connecting each incentre of the incircle of the triangle formed by each excentre and the corresponding side of the triangle with the opposite vertex of ABC, are similarly concurrent. For an online sketch go to: http://dynamicmathematicslearning.com/devilliierspoints.html

Much to my surprise I also discovered in 2008 that these two points of concurrency had been named the 'De Villiers' points by other mathematicians.]