Published in *The Mathematical Gazette*, 98(542), July 2014, pp. 334-339.

A variation of Miquel's theorem and its generalization

MICHAEL DE VILLIERS

An interesting, elementary theorem first stated by Wallace in 1799 and proved by Miquel in 1838 can be stated as follows (see Figure 1): "If on the sides of ΔABC points are selected on each side of a triangle and the circumcircles determined by each vertex and the points on the adjacent sides are constructed, then 1) these circumcircles pass through a common point M; 2) the circumcentres of the Miquel circles form a triangle similar to the original triangle; 3) the lines from M to the 3 initial points on the sides are equi-angular to the sides." The result also holds if the points are chosen on the extensions of the sides (and is most easily proven, generally, using directed angles).

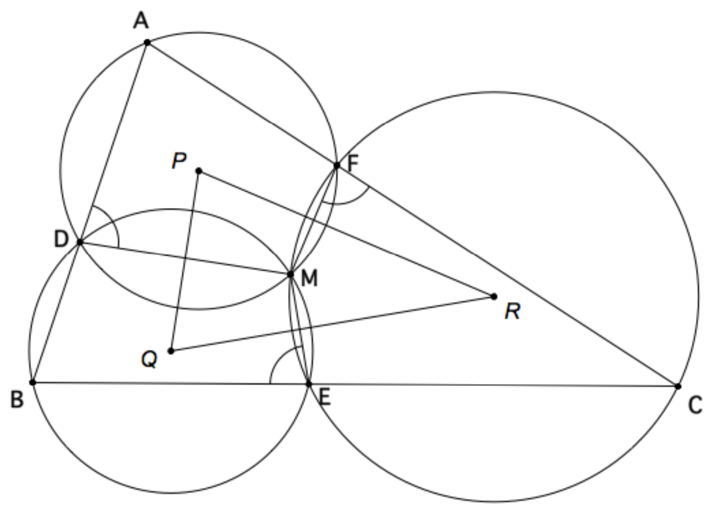

Figure 1

It provides a nice result for high school students to explore with dynamic geometry and is easy to prove with basic cyclic quadrilateral geometry. It appears in several geometry books such as [1], [2], [3] and [4] (where it is called the Pivot Theorem). However, [5] and [6] appear to be the only books that mention as converses the following variation of the Miquel theorem, namely, "If from a point M, equi-angular lines are drawn to the sides of ΔABC to intersect the sides respectively at D, E and F (see Figure 1), then 1) $ADMF$, $BEMD$ and $CFME$ are cyclic; 2) their circumcircles are concurrent at M, 3) their circumcentres form a triangle similar to the original triangle."

The proof is straightforward, almost exactly like the original using the properties of cyclic quadrilaterals, but in reverse, and is left to the reader. This variation is probably seldom stated in the literature because it seems so trivial and simple to prove, and on the surface may even appear merely like a restatement of the theorem.

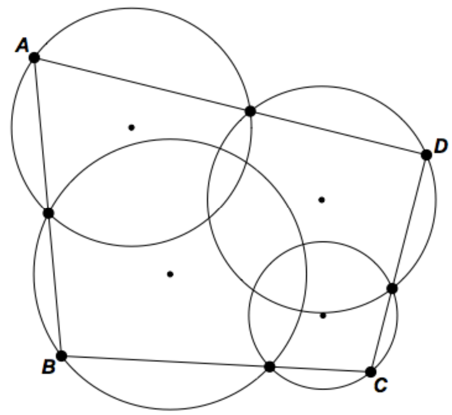

Figure 2

Of interest though is that despite the Miquel theorem not generalizing to polygons as shown in Figure 2, this variation of starting with point *M* generalizes to any polygon as shall now be demonstrated. The reader is also invited to interactively explore the dynamic geometry sketch illustrating the result at: http://www.dynamicmathematicslearning.com/miquel-variation.html

Firstly, consider a quadrilateral *ABCD* with a point *M*, from which equi-angular lines are drawn to meet the sides *AB*, *BC*, *CD* and *DA* respectively in *E*, *F*, *G* and *H* (see Figure 5, which shows the case for perpendicular lines to the sides, and Figure 4, which shows the general case). Then 1) *AEMH*, *BFME*, *CGMF* and *DHMG* are cyclic; 2) the Miquel circles are concurrent at *M*; 3) their circumcentres form a quadrilateral *PQRS* similar to *ABCD*.

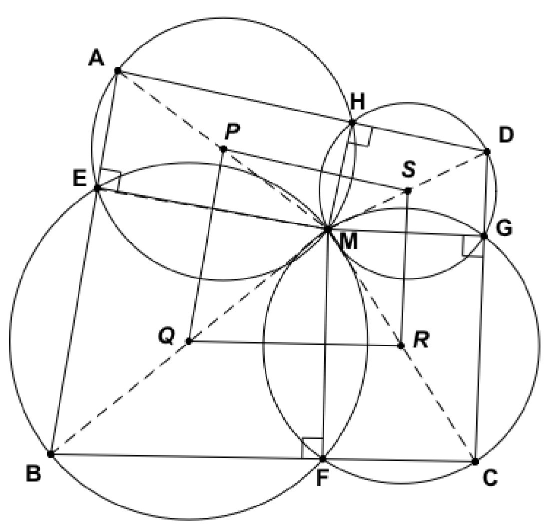

Figure 3

Published in *The Mathematical Gazette*, 98(542), July 2014, pp. 334-339.

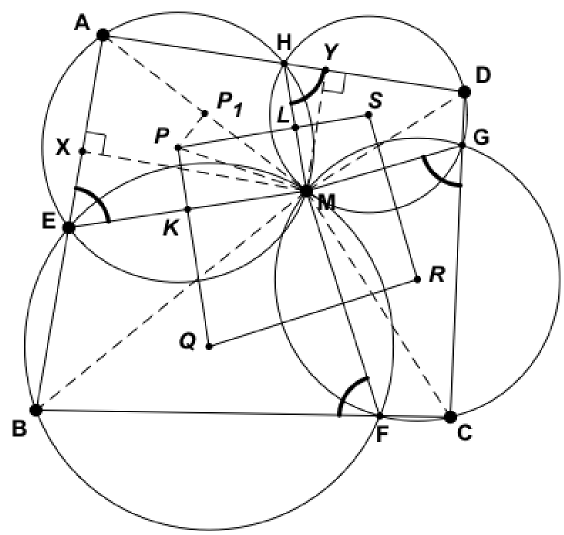

Figure 4

Exactly like the Miquel theorem and the variation for triangles mentioned above, it's easy to see in the general case in Figure 3 that *PQRS* is equi-angular to *ABCD*. For example, since the angles at *K* and *L* are right angles, it follows that *PKML* is also cyclic. Hence, $\angle KPL = \angle EAH$, since both are respectively supplementary to $\angle KML$. In the same way, can easily be shown that the angles at *Q*, *R* and *S* are respectively equal to the angles at *B*, *C* and *D*; and hence that *PQRS* is equi-angular to *ABCD*.

But unlike the case for triangles, equi-angularity is not sufficient to show similarity between two quadrilaterals. So recently preparing a talk and workshop for the Association of Mathematics Education of South Africa (AMESA) Congress in Cape Town in June 2013, which referred among others to this result, I wanted to show the audience that though these two quadrilaterals were equi-angular, they were not necessarily similar. Much to my surprise I found that when doing an accurate dynamic geometry sketch, and then measuring the ratio of the sides between the two quadrilaterals, that in fact the sides were in constant ratio; and hence that they were indeed similar!

Proof

Let us first consider the special case shown in Figure 3 where the lines are drawn perpendicular to the sides. Since the angles at *E*, *F*, *G* and *H* are right angles, it follows that *AEMH*, *BFME*, *CGMF* and *DHMG* are all cyclic, and since *M* is common to all them, their circumcircles are obviously concurrent. Note further that the chords *AM*, *BM*, *CM* and *DM* are all diameters of their respective Miquel circles. It therefore follows that quadrilateral

Published in *The Mathematical Gazette*, 98(542), July 2014, pp. 334-339.

PQRS can be obtained from quadrilateral *ABCD* by a dilation from centre *M* with a scale factor ½; and hence *PQRS* is similar to *ABCD*. (Note: this special case essentially generalizes to a general quadrilateral, the corresponding result for a cyclic quadrilateral with perpendicular bisectors of sides that Chris Pritchard mentioned recently in a talk at the University of KwaZulu-Natal – see [7].)

Let us now consider the general case shown in Figure 4, where P_1 shows the position of *P* when the lines are drawn perpendicular to the sides as in Figure 2. Since the angles at *E*, *F*, *G* and *H* are all equal, it follows that *AEMH*, *BFME*, *CGMF* and *DHMG* are all cyclic, since their respective exterior angles at these points are all equal to the opposite interior angles. Hence, the circumcircles are concurrent at the common point *M*.

Next note that *P* lies on the perpendicular bisector of *AM* since $PA = PM$; hence $\angle PP_1M = 90°$. Right triangles *XEM* and *YHM* are similar since $\angle XEM = \angle YHM$. Since P_1 is determined by the intersection of the perpendicular bisectors of *XM* and *YM*, the same spiral similarity by $\angle XME$ that maps P_1 to *P* will map *XM* to *EM* with scale factor *EM*/*XM*, *YM* to *HM* and P_1M to *PM* with the result that right triangle P_1PM is similar to *XEM* and *YHM*. In the same way for the other points *Q*, *R* and *S* can be shown that the same spiral similarity exists. It therefore follows that *PQRS* is similar to $P_1Q_1R_1S_1$, and hence to *ABCD*. This completes the proof.

Further generalization and a related result

As can be seen in the proof above, the number of vertices is irrelevant, and the result easily generalizes in the same way to any polygon.

Though this variation of Miquel, and specifically its generalization to higher polygons discussed here is probably not original, I have so far failed to find mention of it in the literature surveyed. It seems interesting enough to deserve to be better known.

Acknowledgement: The author wishes to thank the Editor and the Reviewer for valuable comments in preparing and refining this manuscript for publication.

Note: Zipped *Sketchpad* 4 and 5 sketches illustrating the main Miquel variation result discussed in this paper is available for downloading at:
http://dynamicmathematicslearning.com/miquel-converse.zip

References

1. R.A Johnson, *Advanced Euclidean Geometry (Modern Geometry)*, New York: Dover Publications, (1929), pp. 131-135.

2. G. Leversha, *The Geometry of the Triangle*, UKMT, (2013).
3. A.S. Posamentier, *Advanced Euclidean Geometry*, Emeryville: Key College Publishing, (2002), pp. 101-105.
4. D. Wells, *Penguin Dictionary of Curious and Interesting Geometry*, London: Penguin Books, (1991), p. 184.
5. M. de Villiers, *Rethinking Proof with Sketchpad*, Emeryville: Key Curriculum Press, (1999 & 2003), pp. 122-125, 197-198.
6. R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Washington: The Mathematical Association of America, (1995), p. 83.
7. C. Pritchard, 'A tour around the geometry of a cyclic quadrilateral', talk given at the University of KwaZulu-Natal (Edgewood Campus), 12 April 2013, Slide 6. (PowerPoint file of talk is available to view or download at: http://www.scribd.com/doc/148038647/A-tour-around-the-geometry-of-a-cyclic-quadrilateral)

MICHAEL DE VILLIERS

Dept. of Mathematics & Computer Science Education,

University of KwaZulu-Natal, South Africa

e-mail: *profmd@mweb.co.za*

Homepage: http://dynamicmathematicslearning.com/homepage4.html

Dynamic Geometry Sketches:

http://dynamicmathematicslearning.com/JavaGSPLinks.htm