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13

Defining in Geometry

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DEFINITIONS are important in mathematics. They are tools for communication, for reorganizing old knowledge, and for building new knowledge through proof. But definitions present many challenges to both learners and teachers. Mathematical definitions are very concise, contain technical terms, and require an immediate synthesis into a sound concept image. Too often, however, definitions are presented before students' concepts have evolved naturally from existing knowledge. Consequently, students often resort to meaningless memorization. Over time many educators have thought about better ways to teach defining.

Early in the twentieth century the German mathematician Felix Klein strongly opposed the practice of presenting mathematical topics as completed axiomatic-deductive systems. He argued instead for the *biogenetic* principle in teaching, which has also been advocated by Wittmann (1973), Pólya (1981), Freudenthal (1973), and many others. The biogenetic approach takes the stand that the student should retrace, at least in part, the path followed by the original discoverers or inventors, or alternatively, retrace a path by which knowledge could have been discovered or invented. Human (1978, p. 20) calls this the *reconstructive approach*:

> With this term [reconstructive] we want to indicate that content is not directly introduced to pupils (as finished products of mathematical activity), but that

> the content is newly reconstructed during teaching in a typical mathematical manner by the teacher and/or the pupils [freely translated from Afrikaans].

The reconstructive approach allows students to *participate actively* in the development of the content and the related mathematical processes of defining, axiomatizing, conjecturing, and proving. Content is not presented as a finished, prefabricated product, but rather the teacher focuses on the processes by which the content can be developed or reconstructed. Note that a reconstructive approach does not necessarily imply learning by discovery, for it may just involve a reconstructive explanation by the teacher or the textbook.

Mathematicians and mathematics educators alike have often criticized the direct teaching of geometry definitions with no emphasis on the underlying process of defining. For example, as early as 1908 Benchara Blandford (quoted in Griffiths and Howson [1974, pp. 216–17]) wrote,

> To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions, is both interesting and highly educational.

Hans Freudenthal (1973, pp. 417–18) strongly criticized the practice of directly providing geometric definitions on similar grounds. Ohtani (1996, p. 81) argued that the practice of simply telling definitions to students functions to justify the teacher's control over students, to attain a degree of uniformity, to avoid having to deal with students' ideas, and to circumvent problematic interactions with students. Vinner (1991) and others including Battista (2009) have presented arguments and empirical data supporting the thesis that knowing the definition of a concept does not guarantee understanding. For example, students who have been told, and are able to recite, the standard definition of a parallelogram may still not consider rectangles, squares, and rhombi as parallelograms if their concept image (i.e, their private mental picture) of a parallelogram is one in which not all angles or sides are allowed to be equal.

Students often meet mathematics only through the structure presented in formal mathematics textbooks. This structured approach can lead to a common, but false conception that only one correct definition exists for each defined object in mathematics. In fact, several different, correct definitions may exist for a particular concept, so we have a certain amount of freedom in our choice of definitions. Thus definitions can be considered "arbitrary" (Linchevsky, Vinner, and Karsenty 1992, p. 48). For example, a rectangle may be defined as a parallelogram with a right angle, a quadrilateral with three right angles, or a quadrilateral

with lines of symmetry through opposite sides. But too often textbooks give the impression that a rectangle can and must be defined only one way.

Furthermore, the formal approach promotes the misconception that mathematics always starts with a definition, and that definitions of mathematical objects are given *a priori*. Students then do not realize that definitions are not discoveries, but human "inventions." Students rarely understand that a main purpose of definitions is to promote accurate mathematical communication. (See also Blair and Canada [2009] regarding developing definitions with students.)

The ideas shared in this article stem from years of consideration of issues related to mathematical definitions and to a variety of efforts to engage secondary school students and prospective teachers in meaningful work in this domain (de Villiers 1986, 1994, 1998, 2003, 2004). If teachers are to engage students in a reconstructive approach in which they create and critique their own definitions, then teachers themselves must first understand the subtle distinctions among various types of definitions. Those distinctions are the major topic of this article.

We first consider the distinction between partitional and hierarchical methods of classification. Next we examine the characteristics of a correct definition based on necessary and sufficient conditions. Then we look at economical definitions and develop criteria for good definitions. Finally, we discuss how these distinctions can help teachers plan and implement strategies to develop their students' mathematical thinking.

Types of Definitions

Partitional and Hierarchical Definitions

A mutual relationship exists between classifying and defining. The classification of any set of concepts implicitly or explicitly involves defining the concepts involved, whereas defining concepts in a certain way automatically involves their classification. For example, defining a rectangle as *a quadrilateral with two axes of symmetry through opposite sides* will then imply that a square is classified as a special rectangle. However, if a rectangle is defined as *a quadrilateral with two axes of symmetry through opposite sides, but no other lines of symmetry,* then squares are clearly excluded from the set of rectangles.

We describe a definition such as the first as *hierarchical* (i.e., inclusive) because it allows the inclusion of more particular concepts as subsets of the more general concept. The latter we call a *partitional* (i.e., exclusive) definition because the concepts involved are considered disjoint from each other (i.e., squares are not considered rectangles).

If students are given the opportunity to create definitions of their own for such geometric concepts as the quadrilaterals, the result can be a lively and fruitful class discussion of why hierarchical definitions are generally preferred

in mathematics. De Villiers (1994, p. 15) identifies the following important advantages of hierarchical classification over partitional classification. A hierarchical definition—

- leads to more economical definitions of concepts and formulations of theorems,
- simplifies the deductive system and the derivation of properties of more special concepts,
- often provides a useful conceptual schema during problem solving,
- sometimes suggests alternative definitions and new propositions, and
- provides a useful global perspective.

Nevertheless, in some situations we need partitional definitions to distinguish concepts clearly. For example, we have little choice but to create a partitional classification of convex and concave quadrilaterals, because it is not meaningful to view a concave quadrilateral as a special kind of convex quadrilateral, or vice versa (de Villiers 1994).

Furthermore, from a historical perspective, hierarchical definitions have not always been favored. For example, in Book 1 of *Elements,* Euclid partitioned quadrilaterals into five mutually exclusive categories: *square* (both equilateral and right-angled), *oblong* (right-angled but not equilateral), *rhombus* (equilateral but not right-angled), *rhomboid* (opposite sides and angles equal to one another but neither equilateral nor right-angled), and *trapezium* (any other quadrilateral). Similarly, Euclid did not consider an equilateral triangle to be a special case of an isosceles triangle.

Some of the challenges that definitions create for students are illustrated in the interviews and experiences with children in grades 9 to 12 over several years reported in de Villiers (1994). Here is one example (*I* = interviewer, *S* = student):

I: If we define a parallelogram as any quadrilateral with opposite sides parallel, can we then say that a rectangle is a parallelogram?

S: Yes, … because a rectangle also has opposite sides parallel.… But I don't like this definition of parallelograms.… I know we are taught this definition at school and that squares and rectangles are parallelograms (pulls face), but I don't like it.…

I: How would you define parallelograms instead?

S: As any quadrilateral with opposite sides parallel, but not all angles equal.

I: What about rhombi then? … Would you say a rhombus is a parallelogram?

S: Hmm … according to my definition, yes, … but I don't like that either.… I would therefore rather say a parallelogram is a quadrilateral with opposite sides parallel, but not all sides or angles equal.

Clearly this eleventh-grade student (who happened to be a top student at his school) had no problem with drawing correct conclusions from given definitions and making hierarchical class inclusions but preferred not to do so. Moreover, this student clearly exhibited the ability to formulate a definition. But the definitions he preferred were exclusive ones. Battista and Clements (1992, p. 63) have similarly reported two cases of students who were able to follow the logic of a hierarchical classification of quadrilaterals of squares and rectangles but had difficulty accepting it. Quite often the origin of this problem can be traced back to the elementary school, and to direct provision of exclusive definitions or static exclusive images, which then become so fossilized in students' minds that by the time they reach the high school, the students are very resistant to change.

Another challenge for students arises when they struggle to define a more general figure from a more specific one. For example, students may say, "A rhombus is a square with not all angles equal." On the one hand, students are trying to define a rhombus hierarchically as a special square. Yet on the other hand, they are ending up with "A rhombus is a quadrilateral with equal sides, but with not all angles equal," which partitions rhombuses from squares. The authors have frequently observed this type of problem and labeled it as an "inverse hierarchical-partition" definition because students are trying to be hierarchical but instead are partitioning. Textbook authors and teachers often use this approach without realizing the conceptual difficulties it creates. For example, consider the following introduction to a rhombus: "The next shape we are going to be looking at is called a *rhombus*. We can think of this figure as a square that has been pulled out of shape." This approach encourages students to view a rhombus incorrectly as a special kind of square instead of viewing a square correctly as a special kind of rhombus.

Correct Definitions

For our students to participate actively in the construction of definitions, they must know what qualifies as a correct definition. A definition that contains conditions (properties) that are both necessary and sufficient is said to be correct. For a condition in a given description to be necessary, it must apply to all elements of the set we want to define. (The concept implies the property, so the property is necessary for the concept.) However, for a condition to be sufficient, it must ensure that whenever it is met, we obtain all the elements of the set we want to define. (The property implies the concept, so the property is sufficient for the concept.)

It is helpful to recall that logically in the biconditional $p \Leftrightarrow q$, the condition p is viewed as necessary and sufficient for the condition q, meaning that one

can conclude that q follows from p, and vice versa. The defining conditions for a set must be both necessary and sufficient. For example, consider the following candidate for a definition: "A rectangle is any quadrilateral with opposite sides parallel." Below are some drawings of a quadrilateral complying with the condition "opposite sides parallel."Certainly the statement does apply to elements of the set we want to define (see fig. 13.1(a).). Therefore we can say that *"opposite sides parallel"* is a necessary condition for a rectangle. However, looking at the three drawings, we notice the existence of elements (figs. 13.1(b) and 13.1(c)) that do not belong to the set we want to define. Thus "opposite sides parallel" is not sufficient to guarantee that a particular quadrilateral is a rectangle. Hence we say that "opposite sides parallel" is a necessary but not a sufficient condition for rectangles.

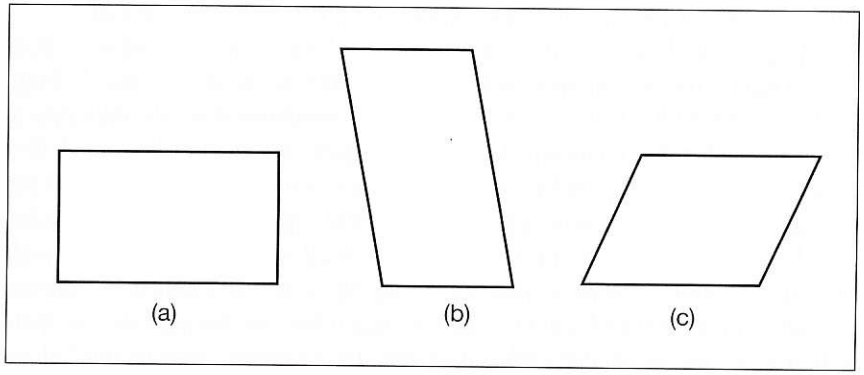

Fig. 13.1. Example and nonexamples of a rectangle

However, having congruent diagonals that are perpendicular bisectors of each other is a sufficient condition for a rectangle (in fact, sufficient for a square). But this condition is not necessary, because it does not apply to many rectangles, including the one in figure 13.1(a).

Next consider the following: "A rectangle is a quadrilateral with opposite sides parallel and with one interior angle equal to 90 degrees." This property applies to every rectangle (making it a necessary condition), and any figure we construct with it will be a rectangle (making it a sufficient condition). Therefore this statement provides a necessary and sufficient condition for rectangles.

Incorrect Definitions

A definition is incorrect if it contains insufficient or unnecessary properties. For example, consider the following:

1. "An isosceles trapezoid is any quadrilateral with perpendicular diagonals."

2. "A kite is a quadrilateral with perpendicular diagonals."

The first statement is incorrect because it contains an unnecessary property, in that isosceles trapezoids do not in general have perpendicular diagonals. The second statement is also an incorrect definition because it does not contain sufficient properties to define a kite and is therefore incomplete. For example, we can construct a diagonal and another one perpendicular to it, and then connect the endpoints, obtaining a quadrilateral as shown in figure 13.2, which is clearly not a kite. In general, to show that a definition is incomplete, it suffices to give a counterexample that meets the purported definition but is not an example of the set of figures one wants to define.

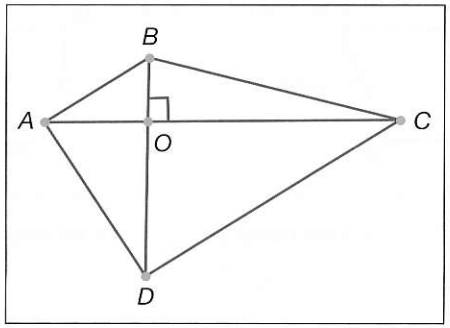

Fig. 13.2. Nonexample of a kite

Note that the statement "A kite is a quadrilateral with perpendicular diagonals" is a correct statement about a property of kites, but it contains too little information to be used as a definition. We therefore say that having "perpendicular diagonals" is a necessary but not a sufficient condition for kites.

The authors have observed that one common difficulty students have in producing correct counterexamples to incomplete definitions is that they often try to refute a definition with a special case. For example, for the incorrect definition "A rectangle is any quadrilateral with congruent diagonals," some students will provide a square as a counterexample. But obviously a square is not a valid counterexample, because a square *is* a rectangle.

Therefore, students should already have developed a sound understanding of a hierarchical (inclusive) classification of quadrilaterals *before* being engaged in formally defining the quadrilaterals themselves (Craine and Rubenstein 1993). This development can be fostered by using interactive geometry software, figures created with flexible wire, or paper-strip models of quadrilaterals. For example, if students use an interactive geometry tool to construct a quadrilateral with opposite sides parallel, then they may notice that they can drag it into the shape of a rectangle, rhombus, or square as shown in figure 13.3. Students can then be asked to describe what this outcome tells them. Teachers should help students realize

that this demonstration shows that rectangles, rhombuses, and squares are special cases of a parallelogram.

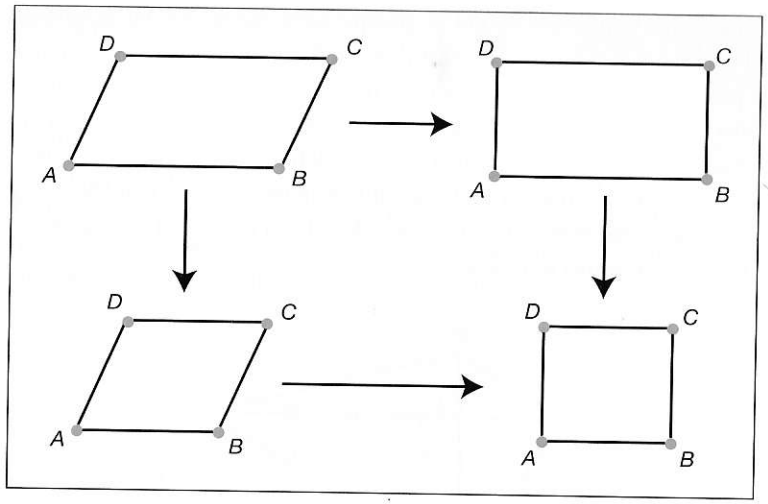

Fig. 13.3. Dynamic transformation of a parallelogram

Economical Definitions

A correct definition can be either economical or uneconomical. An economical definition has a minimal set of necessary and sufficient properties; that is, it has no superfluous information. Conversely, an uneconomical definition contains redundant properties.

For example, consider the following candidates for definitions of a kite:

1. "A kite is a quadrilateral with two pairs of congruent adjacent sides and one pair of opposite angles congruent."
2. "A kite is a quadrilateral with perpendicular diagonals with one being bisected by the other."

The first one is correct, but uneconomical because it contains too much information. In other words, the conditions are necessary and sufficient, but not all of them are required. But which condition can be left out? When students evaluate these conditions, they may realize that if they were to leave out "two pairs of congruent adjacent sides," they would obtain an incorrect definition because it is possible to construct a quadrilateral with one pair of congruent, opposite angles that is not a kite (fig. 13.4).

However, we can construct a kite according to the condition "two pairs of congruent adjacent sides" by placing a compass first at A and then at D and drawing circular arcs as shown in figure 13.5. In addition, we can easily show that this

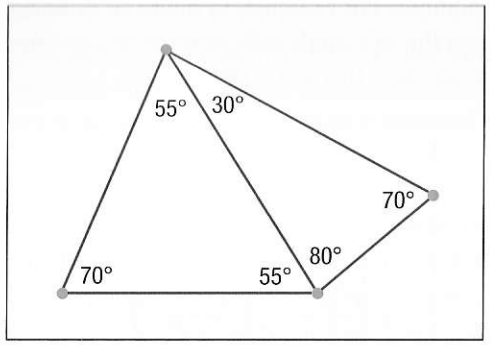

Fig. 13.4. Nonexample of a kite

condition logically implies that "one pair of opposite angles are congruent" because triangles ADB and ADC are clearly congruent (by side-side-side congruence), and therefore $m\angle B = m\angle C$.

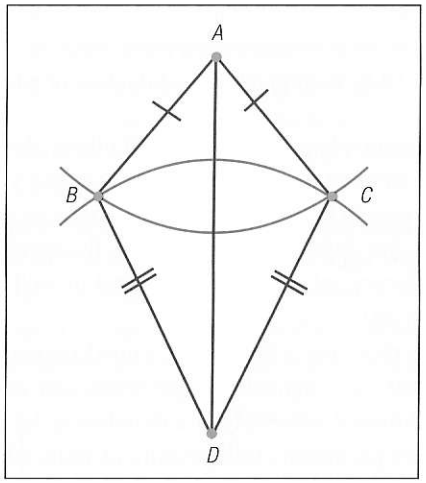

Fig. 13.5. Testing a definition of a kite

The second candidate is a correct, economical definition because no superfluous information is supplied. In other words, we cannot leave out any of the properties, and we can construct a kite from the given conditions. We can demonstrate the latter, as shown in figure 13.6, by first constructing two perpendicular lines, then placing the compass at O to cut off equal distances on the one line, and picking two arbitrary points on the other to determine the vertices. In addition, students can show that the other properties of a kite can be logically deduced

from the given conditions. For example, in figure 13.6, triangles AOB and AOC are clearly congruent (by side-angle-side congruence) and therefore $AB = AC$. In the same way they can show that $DB = DC$.

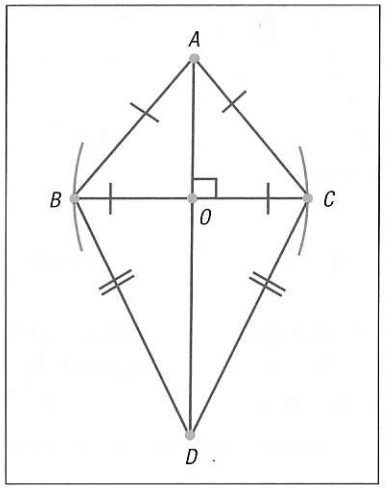

Fig. 13.6. Testing another definition of a kite

This minimality principle—that is, that definitions should be economical—is a crucial structural element of mathematics as a deductive system. As a matter of fact, it shapes the way in which mathematics progresses when it is presented deductively, for after the definition is presented, theorems that give additional information about the concept are formulated and proved (Linchevsky, Vinner, and Karsenty 1992, p. 54).

For example, if a rhombus is defined as a quadrilateral with four congruent sides, then the fact that the diagonals are perpendicular bisectors of each other can be proved as a theorem. Conversely, if a rhombus is defined as a quadrilateral with diagonals that are perpendicular bisectors of each other, then the fact that the four sides are congruent becomes a theorem. Both definitions are economical insofar as the defining conditions contain no superfluous information. This goes back to the idea that definitions are arbitrary. Different sets of definitions produce different theorems within a system.

However, in a few instances in mathematics, definitions are not minimal. A familiar example is the way in which some textbooks define congruent triangles: "Congruent triangles are triangles that have all pairs of corresponding sides congruent and all corresponding angles congruent." We know that it is sufficient to require less than that for two triangles to be congruent, and the fact that less is required is expressed by each of the four postulates normally accepted in high school textbooks for the congruency of triangles.

Convenient Definitions

Obviously a good definition, as we have seen in the foregoing, avoids redundant information; it must be economical. But a good definition also has other characteristics. A good, or convenient, definition is one that also allows us to deduce the other properties of the concept easily; that is, it should be *deductive-economical.*

A valuable exercise for students is to have them compare different definitions according to this criterion. For example, the definition of a rhombus as a quadrilateral with two axes of symmetry through the opposite angles is more deductive-economical than the standard textbook definition of it as a quadrilateral with all sides congruent. For the former, the other properties (e.g., perpendicular, bisecting diagonals, all sides congruent, and so on) follow immediately from symmetry, whereas with the latter, we have to use congruency and somewhat longer arguments to deduce the other properties.

A Teaching Sequence

Young children most easily learn what a table or a chair is by seeing many different examples of those concepts, not by being provided with formal definitions of a table and a chair. (See also Battista [2009] for more about this phenomenon of concept formation.) Similarly, without starting with a formal definition, children can easily learn what a square, rectangle, or kite is.

In an interactive geometry environment, concepts such as the special quadrilaterals can be introduced in three stages. The first stage uses the software to help students learn what a specific shape is (a concept image). For example, a concept such as an isosceles trapezoid can easily be introduced as shown in figure 13.7 by first having students construct a line AB, then construct a line segment CD, and

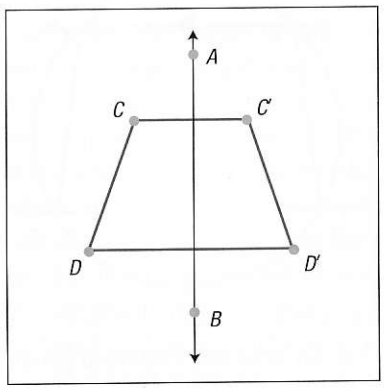

Fig. 13.7. Introducing an isosceles trapezoid by construction

subsequently reflect CD in the line AB to obtain $C'D'$. By connecting vertices, students can then be asked to explore the properties of quadrilateral $CDD'C'$, which they may be told is called an *isosceles trapezoid* (compare de Villiers [2003]).

By dragging the figure and measuring its attributes, students can develop a sound concept image of an isosceles trapezoid as a quadrilateral having many properties, including congruent diagonals, two pairs of congruent adjacent angles, at least one pair of opposite sides congruent, at least one pair of opposite sides parallel, and others. Moreover, students can discover that they can drag an isosceles trapezoid into the shape of a rectangle and square, but not a general rhombus, parallelogram, or kite, thus forming the foundation for a hierarchical view.

The second stage involves challenging students to write their own correct, economical definitions for an isosceles trapezoid, and then to test those definitions by means of construction and measurement. Equivalently, students can be challenged to devise different ways of constructing a dynamic isosceles trapezoid that always remains an isosceles trapezoid no matter how it is dragged. Such constructions, as reported in Smith (1940), help develop an understanding not only of the difference between a *premise* and a *conclusion* but also of the *causal* relationship between them, that the conditions force the result. For example, consider a circle with two parallel lines intersecting it (see fig. 13.8). Then the quadrilateral formed by the points where these two parallel lines intersect the circle must have congruent diagonals and congruent opposite sides and, hence, is an isosceles trapezoid. This result shows that the condition is sufficient to ensure an isosceles trapezoid, and that we could define an isosceles trapezoid as any cyclic quadrilateral with at least one pair of opposite sides parallel.

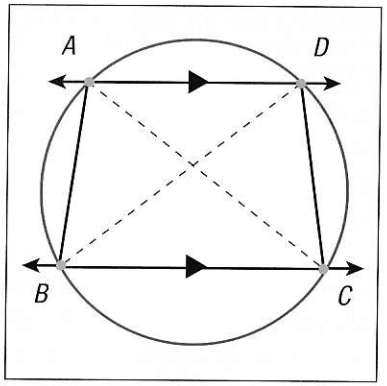

Fig. 13.8. An isosceles trapezoid constructed from two parallel lines intersecting a circle

The third stage involves the formal systematization of the properties of an isosceles trapezoid. For example, by starting from any given definition, students then have to deduce the other properties logically from it as theorems. A popular initial choice suggested by students, teachers, and some textbooks is the following: "An isosceles trapezoid is any quadrilateral in which at least one pair of sides are parallel and the other pair of opposite sides are congruent." However, this definition is *incorrect*. For example, although the conditions can produce an isosceles trapezoid, the conditions can also produce a parallelogram. Students need to realize that a general parallelogram cannot be considered an isosceles trapezoid, as parallelograms do not necessarily have all properties of an isosceles trapezoid (e.g., congruent diagonals, two pairs of adjacent angles congruent, cyclic, axis of symmetry, and so forth).

Although students usually attempt to improve this definition, after a while they find no satisfactory way of correcting it. If they formulate it in such a way as to exclude the parallelograms—for example "An isosceles trapezoid is any quadrilateral with one pair of opposite sides parallel, and another pair of opposite sides equal but not parallel"—they exclude not only the (general) parallelogram but also rectangles and squares.

After further discussion and critical comparison of various correct, economical definitions for an isosceles trapezoid, students usually settle with a convenient definition, such as "an isosceles trapezoid is any quadrilateral with an axis of symmetry through a pair of opposite sides." This definition is much easier to use to derive other properties. Contrast it, for example, with a definition such as "an isosceles trapezoid is any cyclic quadrilateral with at least one pair of opposite sides parallel."

Conclusion

Students should be given the opportunity to engage in the process of constructing definitions. Interactive geometry software is a tool that can help promote this goal (e.g., de Villiers [2003]). It is plausible to conjecture that, through experiences in which definitions are not supplied directly, students' understanding of geometric definitions, and of the concepts to which they relate, will increase. Furthermore, students are likely to develop a better understanding of the nature of definitions as well as skill in defining objects on their own. In particular, they may come to realize that definitions should be economical and that they are a matter of choice. Recent results from Govender and de Villiers (2002), de Villiers (1998, 2004), and Sáenz-Ludlow and Athanasopoulou (2007) do indicate some improvement and positive gains in students' understanding of the nature of definitions, as well as in their ability to define geometric concepts themselves.

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A Sketchpad 4 file for "Exploring the Hierarchical Classification of Quadrilaterals" is available at:
http://dynamicmathematicslearning.com/quad-classify.gsp

An interactive "Hierarchical Classification of Quadrilaterals" is available online at:
http://dynamicmathematicslearning.com/quad-tree-web.html

Geometer's Sketchpad files that support this article are found on the CD-ROM disk accompanying this Yearbook.