Reflecting on a 2nd Round 2013 SA Mathematics Olympiad Problem

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The following problem was used as Question 13 in the 2nd Round of the 2013 Senior South African Mathematics Olympiad (SAMO).

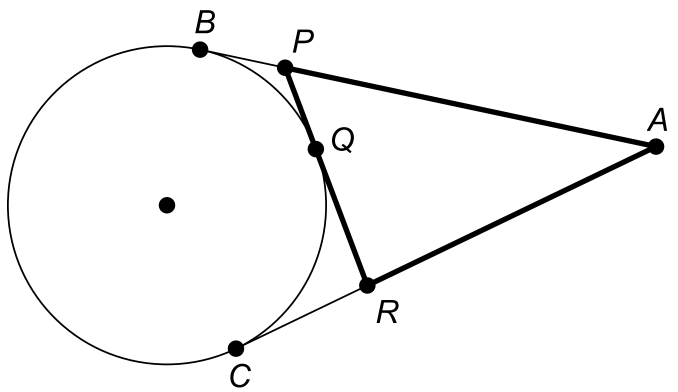

FIGURE 1

Two tangents are drawn to a circle from a point A, which lies outside the circle as shown in Figure 1. The two tangents touch the circle at points B and C respectively. A third tangent intersects AB in P and AC in R, and touches the circle at Q. If $AB = 20$, find the perimeter of triangle APR.

Solution

If we let $PQ = x$, then from the 'tangents from a point to circle' theorem, $PB = x$, and $AP = 20 - x$. Similarly, if we let $QR = y$, we have $CR = y$, and $RA = 20 - y$. So the perimeter of triangle APR is:

$$AP + PQ + QR + RA = (20 - x) + x + y + (20 - y) = 40$$

Quite surprisingly, and perhaps even counter-intuitively, it turns out that the perimeter of triangle APR is only dependent on the length of AB and completely *independent* of not only the position and length of PR, but also the size of the circle. So the problem turns out to be based on an interesting underlying theorem. The reader is now invited to experience this first hand by exploring a dynamic, interactive sketch of this theorem online, by varying PR and the size of the circle by dragging its centre at:

http://dynamicmathematicslearning.com/tangent-perimeter-triangle-theorem.html

Of course, a clever student who has had a lot of experience of Mathematical Olympiads might immediately realize, from the given diagram (Figure 1), that since no lengths of PR are given to fix it, one may simply move it until P coincides with B, in which case R coincides with A. It is then immediately obvious that the constant perimeter of triangle APR is $2\times AB = 40$. (Or alternatively, shrink the circle to zero radius). To prevent the few students who might elegantly reason like this, it was then decided by the Olympiad committee to rather let $PQ = 3$; hence making it perhaps a little easier for more students, but unfortunately somewhat obscuring the underlying theorem.

A nice, direct application of the theorem is demonstrated in Figure 2, which shows two triangles ABC and KLM overlapping, and circumscribed around the same circle. If either one or both of these two triangles are rotated, then the perimeters of the shaded triangles APU, KQP, etc. remain constant (provided none of the points P, Q, R, etc. move onto the extensions (outside) of any of the sides of triangles ABC and KLM).

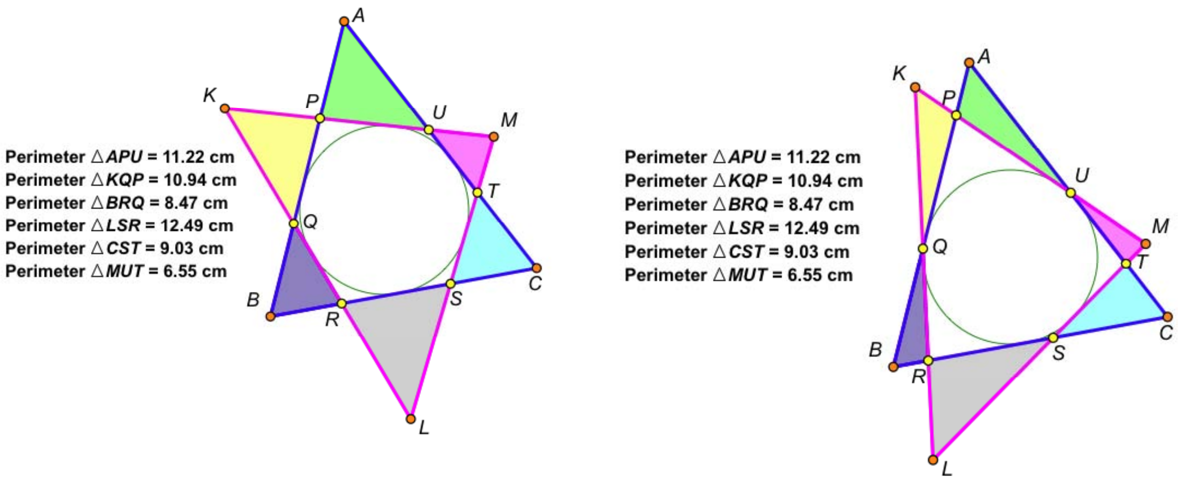

FIGURE 2

The reader is once again encouraged to experience this result dynamically by visiting the URL link given earlier.

Note: Past papers with solutions from 1997 onwards of the Junior and Senior South African Mathematics Olympiad are available at: http://www.samf.ac.za/QuestionPapers.aspx