From Van Zyl, J.V. et al. (2011). South Africa and the 52nd International Mathematical Olympiad, p. 23, pp. 88-90; published by the SA Mathematics Foundation.

32. Consider an $n \times n$ chessboard with $n > 1$, $n \in \mathbb{N}$. In how many different ways can $2n-2$ identical pebbles be placed on the chessboard (each in a different square) such that no two pebbles are on the same diagonal? (Two pebbles are on the same diagonal if the line joining the midpoints of the squares the lie in is parallel to one of the main diagonals of the chessboard.)

33. Let $ABCD$ be a cyclic quadrilateral and let the diagonals AC and BD intersect in E. Let O_1 and I_1 be the circumcentre and incentre, respectively, of triangle ABE and let O_2 and I_2 be the circumcentre and incentre, respectively, of triangle CDE. Prove that

$$(AO_2^2 - BO_2^2) - (AI_2^2 - BI_2^2) = (DO_1^2 - CO_1^2) - (DI_1^2 - CI_1^2).$$

34. Find all $k \in \mathbb{N}$ such that there exists an integer a such that $(a+k)^3 - a^3$ is divisible by 2007.

35. Find all $n \in \mathbb{N}$ such that there exists integers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) - (a_1 b_1 + \cdots + a_n b_n)^2 = n.$$

36. Let Γ be a circle and Γ_1, Γ_2, Γ_3 and Γ_4 be smaller circles with their centres O_1, O_2, O_3 and O_4 respectively. For $i = 1, 2, 3, 4$ and $\Gamma_5 = \Gamma_1$ the circles Γ_i and Γ_{i+1} meet at A_i and B_i. If the points O_1, A_1, O_2, A_2, O_3, A_3, O_4 and A_4 lie on Γ, in that order, and are pairwise different, prove that $B_1B_2B_3B_4$ is a rectangle.

37. Prove that the midpoints of the altitudes of a triangle are collinear if and only if the triangle is right-angled.

38. For $n \geqslant 4$, prove that if $\lfloor \frac{2^n}{n} \rfloor$ is a power of 2, then n is a power of 2.

39. Let T be the set of divisors of 2004^{100}. What is the size of the largest possible subset S of T such that if $x, y \in S$, then x does not divide y?

40. Determine all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$xf(x + xy) = xf(x) + f(x^2)f(y)$$

for all real numbers x and y.

the pebbles on two opposite corner squares of this rectangles, hence on the edge of the board. By the discussion above, this is true for $k = 2$.

So, assume that it is true for all $1 \leqslant k \leqslant m$. In particular this means that there is a pebble on each diagonal of type B within $m-1$ squares of the main diagonal. This means that we cannot place a pebble on any of those $2m-1$ (main + $m-1$ to each side) diagonals. At this point we also have a pebble on each of the $m-1$ diagonals of type A closest to the top left corner (m if we include the one square diagonal in the corner). Now consider step $m+1$ where we consider the next diagonal of type A, i.e. the one that lies m squares away from the corner. The only squares that are still available to put a pebble on are the ones on the edge of the board, i.e. the one m squares below the corner, and the one m squares to the right of it. Pick one of these. Then the pebble that lies m squares away from the bottom right square is also fixed (by symmetry there are only 2 squares available on the diagonal of type A that lies m squares away from the bottom right corner square, and one of these are now unavailable, because of our latest choice) and lies on the opposite corner square of the rectangle.

We are finally ready to do some counting. There are two main diagonals and $n-1$ of these rectangles (corresponding to the $n-1$ diagonals of type A that lie in the upper left triangle of the board). For each rectangle, there are two choices of how to place the pebbles (either this pair of opposite corners, or the other pair), and for each main diagonal there are two ways of placing the pebbles (either the one corner, or the other), hence in total there are 2^{n+1} ways to do this.

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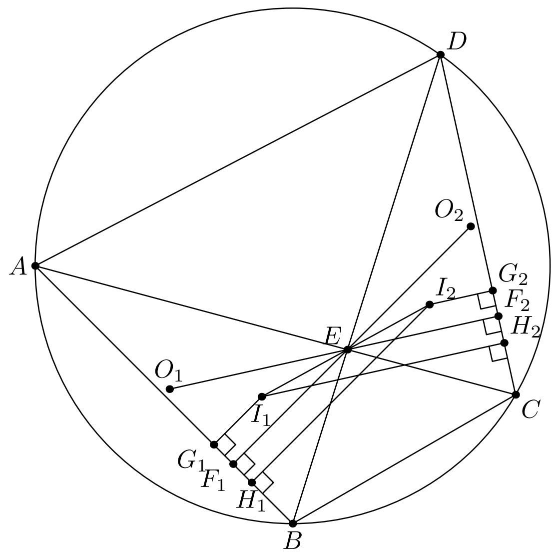
33. Let us first recall two general facts. The first is about differences between squares. Let AB be two points on a line, C a third point in the plane, and D the point on AB such that $CD \perp AB$. Then $AC^2 - BC^2 = (AD^2 + CD^2) - (BD^2 + CD^2) = AD^2 - BD^2$. For the second, let ABC be a triangle, O its circumcentre, and H its orthocentre. Then $\angle BAH = \angle OAC$ and similarly for the other vertices. This can be done by simple angle chasing.

Next, note that, since $ABCD$ is cyclic, we have $\angle CAB = \angle CDB$ and $\angle ABD = \angle ACD$, so that $\triangle ABE \sim \triangle DCE$. Suppose that the ratio of similarity is $x := \frac{DC}{AB}$. Since EI_1 bisects $\angle AEB$, EI_2 bisects $\angle DEC$ and $\angle DEC$ is opposite $\angle AEB$, I_1, E and I_2 are collinear. Next, by our second general fact, we claim that O_2E extended is perpendicular to AB and that O_1E extended is perpendicular to DC. For this suppose that the perpendiculars from E to AB and CD meet AB and CD in F_1 and F_2 respectively. Then $\angle AEF_1 = \angle O_1EB$, and since $\triangle ABE \sim \triangle DCE$, these angles are also equal to $\angle DEF_2 = \angle O_2EC$. It follows that O_1EF_2 and O_2EF_1 are straight lines. Also denote the perpendiculars from I_1 onto AB and DC by G_1 and H_2 respectively and the perpendiculars from I_2 onto AB and CD by H_1 and G_2 respectively.



The left hand side of the equation becomes

$$\begin{aligned} & (AO_2^2 - BO_2^2) - (AI_2^2 - BI_2^2) \\ = \; & (AF_1^2 - BF_1^2) - (AH_1^2 - BH_1^2) \\ = \; & (AF_1 + BF_1)(AF_1 - BF_1) - (AH_1 + BH_1)(AH_1 - BH_1) \\ = \; & AB(AF_1 - AH_1 + BH_1 - BF_1) \\ = \; & 2AB \cdot H_1F_1, \end{aligned}$$

using directed segments. Similarly the right hand side becomes

$$(DO_1^2 - CO_1^2) - (DI_1^2 - CI_1^2) = 2DC \cdot H_2F_2.$$

Note that the quadrilaterals $EI_1G_1F_1$ and $EI_2G_2F_2$ are similarly constructed inside similar triangles ABE and DCE, so they are similar to each other with the same factor of similarity. Therefore $\frac{F_2G_2}{F_1G_1} = x$. Now note that $\frac{I_2E}{EI_1} = x$ as well, since they correspond to the same construction in similar triangles. Therefore their projections onto the lines AB and CD must be in the same ratio. So $\frac{H_2F_2}{F_2G_2} = \frac{I_1E}{EI_2} = 1/x$ and $\frac{H_1F_1}{F_1G_1} = \frac{I_2E}{EI_1} = x$. Finally

$$\frac{H_2F_2}{H_1F_1} = \frac{H_2F_2}{F_2G_2}\frac{F_2G_2}{F_1G_1}\frac{F_1G_1}{H_1F_1} = \frac{1}{x}\cdot x \cdot \frac{1}{x} = \frac{1}{x} = \frac{AB}{DC},$$

implying that $2AB \cdot H_1F_1 = 2DC \cdot H_2F_2$, which is what we wanted.

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34. Note that $2007 = 3^2 223$, where 223 is prime and that $(a+k)^3 - a^3 = 3k^3 + 3ak^2 + a^2k$. If k is not divisible by 3, then a^2k has to be divisible by 3 (since $3|3k^3 + 3ak^2 + a^2k$), which implies that a is divisible by 3. But then $3ak^2 + a^2k$ is divisible by 9, but $3k^3$ is not. So, it is necessary that $k = 3m$ for some integer m. We claim that this is also sufficient. Substituting $k = 3m$ and $a = 3b$ in the equation tells us that $3k^3+3ak^2+a^2k = 27(3m^3+3bm^2+b^2m) = 27m(3m^2+3bm+b^2)$ must be divisible by 223.

If 223 also divides k, then we may take $a = 0$, since k^3 is divisible by 223 and by 9. So suppose that k, and hence m is not divisible by 223. Since 223 is prime, this means that m has a multiplicative inverse modulo 223, or alternatively there exists some integer c such that $cm \equiv 1 \pmod{223}$. If 223 divides $27m(3m^2 + 3bm + b^2)$, then it also divides $3m^2+3bm+b^2$ (since 223 does not divide $27m$) and hence it also divides $c^2(3m^2+3bm+b^2) = 3c^2m^2+3bmc^2+b^2c^2 \equiv 3+3bc+(bc)^2$.

So, if we can find some x such that $3 + 3x + x^2$ is divisible by 223, then we can set $b = mx \equiv c^{-1}x \pmod{223}$, where the inverse is taken modulo 223 and this will give a solution to the original problem. To show that $3+3x+x^2$ has a root modulo 223, write it as $x^2+3x+3 \equiv x^2-220x+3 = (x-110)^2+3-110^2 \equiv (x-110)^2-55 \pmod{223}$. Using quadratic reciprocity, we show that 55 is a quadratic residue modulo 223:

$$\begin{aligned}
\left(\frac{55}{223}\right) &= \left(\frac{5}{223}\right)\left(\frac{11}{223}\right)\\
&= \left(\frac{223}{5}\right) \times \left[-\left(\frac{223}{11}\right)\right]\\
&= -\left(\frac{3}{5}\right)\left(\frac{3}{11}\right)\\
&= -(-1) \times -\left(\frac{11}{3}\right)\\
&= 1,
\end{aligned}$$