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A Multiple Solution Task: a SA Mathematics Olympiad Problem

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Exploring and utilizing multiple solution tasks (MST), where students are given rich mathematical tasks and encouraged to find multiple solutions (proofs), has been an interesting new and productive trend in problem-solving research in recent years. A longitudinal, comparative study by Levav-Waynberg and Leikin (2012) indicated that an MST approach in the classroom provided greater opportunity for potentially creative students to present their creative products than the conventional learning environment.

Reflection and discussion in the classroom of MST problems can be a very useful learning experience for developing young mathematicians. George Polya (1945) specifically mentions the consideration of alternative solutions or proofs in the final stage of his model for problem solving, namely 'Looking Back'. The value of this process of reflection is not only to better understand the problem by viewing it from different perspectives, but also to encourage flexibility of thought and to enhance one's repertoire of problem solving skills and approaches for future challenges. Unfortunately in most traditional classrooms very little such opportunity for mathematically talented learners are usually created.

Although the South African Mathematics Olympiad (SAMO) doesn't explicitly ask students to arrive at multiple solutions for problems (nor does it provide an opportunity for students to write down their full solutions in the first two rounds), there are usually several problems in the first two rounds that can be solved and proved in numerous ways. In fact the potential for a problem to be solved in multiple ways is frequently used as a selection criterion, especially for harder problems.

We will now look at one such MST example from the 1st Round of the 2016 Senior South African Mathematics Olympiad[1].

[1] Past papers of the SA Mathematics Olympiad are available at: http://www.samf.ac.za/QuestionPapers.aspx

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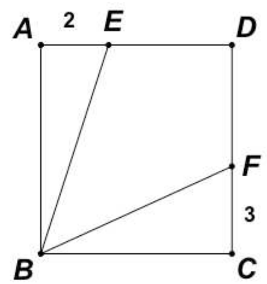
The Problem



If $ABCD$ is a square, $\angle EBF = \angle CBF$, $AE = 2$ and $CF = 3$, then the length of EB is

(A) $\sqrt{13}$ (B) 5 (C) 6 (D) $\sqrt{29}$ (E) $4\sqrt{3}$

Actually underlying this problem is the more general, interesting theorem, which can be stated as follows: If $ABCD$ is a square, and an arbitrary point E is chosen on AD and $\angle EBC$ is bisected by ray BF, with F on CD, then $EB = AE + CF$. Before reading further, readers are encouraged to first dynamically explore the general theorem at the interactive sketch at http://dynamicmathematicslearning.com/samo2016-R1Q20.html as well as attempting to prove the result themselves.

Multiple Solutions and Proofs of this Result

Three official solutions were given for Question 20, namely, 1) trigonometric, 2) similar triangles, and 3) by a 90° rotation.

1) If we let $\angle FBC = \theta$, then $\angle EBF = \theta$ also, and $\angle AEB = 2\theta$ (alternate angles). If x denotes the side of the square, then $\tan\theta = 3/x$ from triangle BCF and $\tan 2\theta = x/2$ from triangle BEA. From the double-angle formulae $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ it is easy to show that if $t = \tan\theta$, then $\tan 2\theta = 2t/(1 - t^2)$.

Thus we have $\dfrac{x}{2} = \dfrac{6/x}{1-(3/x)^2} = \dfrac{6x}{x^2-9}$.

This simplifies to $x^2 - 9 = 12$ (since $x \neq 0$), so $x^2 = 21$. Finally, by Pythagoras' theorem $BE^2 = AB^2 + AE^2 = x^2 + 4 = 21 + 4 = 25$, so $BE = 5$.

2) Alternatively, it is possible to solve the problem using only similar triangles. Let G be the point on EB such that $EG = EA$ and let H be the foot of the perpendicular from G to AB. Then triangles BHG and BAE are similar. Also $\angle EAG = 90 - \theta$, since triangle EAG is

isosceles, so $\angle GAH = \theta$ and triangle AHG is similar to triangle BCF. Using these facts, we have

$\frac{GB}{EG} = \frac{HB}{AH} = \frac{HB}{HG} \cdot \frac{HG}{AH} = \frac{AB}{AE} \cdot \frac{CF}{BC} = \frac{CF}{AE}$, since $AB = BC$.

It follows that $GB = CF$, since $EG = AE$ by construction, and finally $EB = EG + GB = AE + CF = 5$, as required.

3) An even quicker and more elegant solution is as follows. Rotate triangle ABE through 90° clockwise around B, so that A coincides with C and E is at position E' on DC produced. (In other words, let E' be the point on DC produced such that $CE' = AE$, and join B and E' so that triangles BAE and BCE' are congruent.) Then $\angle E'FB = 90 - \theta$ and $\angle E'BF = \angle E'BC + \angle CBF = (90 - 2\theta) + \theta = 90 - \theta$. Thus triangle $E'BF$ is isosceles, and therefore $E'B = E'F$. Since $E'B = EB$ and $E'F = E'C + CF = AE + CF$ it follows that $EB = AE + CF = 5$.

After the paper was written, the SAMO committee also received the following neat, alternative solution by Nicholas Kroon in Grade 12 at St. Andrew's College in Grahamstown:

4) Let $\angle EBF = \angle FBC = \theta$. Let x denote the side of the square. We will calculate the area of the square in 2 different ways.

$$[ABCD] = x^2 = [BAE] + [BFC] + [EDF] + [BEF]$$

Using Pythagoras and sine area rule, we have

$$x^2 = x + \frac{3}{2}x + \frac{(x-2)(x-3)}{2} + \frac{1}{2}\sqrt{x^2+4} \times \sqrt{x^2+9} \times \sin\theta$$

Note from ΔBFC we know that $\sin\theta = \frac{3}{\sqrt{x^2+9}}$. Hence

$$x^2 = \frac{5}{2}x + \frac{(x-2)(x-3)}{2} + \frac{1}{2}\sqrt{x^2+4} \times 3$$

$$\Rightarrow 2x^2 = x^2 + 6 + 3\sqrt{x^2+4}$$

$$\Rightarrow (x^2-6)^2 = 9x^2 + 36$$

$$\Rightarrow x^2(x^2-21) = 0$$

Note since $x > 0$, it follows that $x = \sqrt{21}$ and hence $EB = 5$ by Pythagoras.

It is left to the reader to compare the pros and cons of each solution, but in particular, note that the general theorem for a square and an arbitrarily chose point E on AD is most easily seen and immediately generalisable from proofs 2 and 3. It can also be seen from proof 3 that attempting a similar rotation of triangle ABE for a rectangle or a rhombus won't preserve the same general relationship $EB = AE + CF$. Finally, though the approaches from proofs 1 and 4 are different, both arrive at basically the same equation, and is probably the sort of approach more likely to be favored by the algebraically inclined.

It would be wonderful if more pupils, and also teachers, submitted interesting alternative solutions for SAMO questions, as it is likely that the official solutions are often not the only ones.

References

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