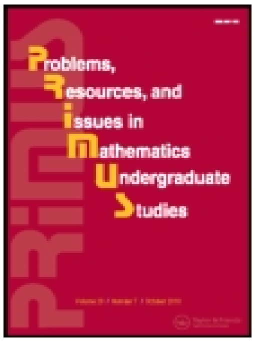

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THE KINEMATIC METHOD IN GEOMETRY

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ABSTRACT: We present and discuss the use of arguments of velocity to demonstrate invariance of relationships among segments of geometrical figures.

KEYWORDS: Kinematic method, velocity of endpoints, vectors, geometry, dynamic geometry programs.

INTRODUCTION

The capacity of dynamic geometry programs [5] to trace the locus of points by drawing the position of a point at successive times provides a powerful help to see how the velocities of endpoints of segments are related. With the *Trace Discretely* option on, when the point is dragged quickly, the separation between the points that form its trace will be bigger than when the point is dragged slowly. In Figure 1, C is the midpoint of segment AB. When A is stationary, and AB is stretched and rotated, the magnitude of the velocity of C is half that of B as both move simultaneously. The separation between successive positions of B is twice as big as that of C. The total distance traveled by B will also be twice that of C. In this case the two trajectories are similar, with a dilation factor of 2.

Dynamic geometry programs provide a very convenient setting for students to understand and use the kinematic method, in particular, the theory of velocities. In this paper we will consider the points of geometrical figures as endpoints of changing vectors. We will think of segments that are deformed as changing vectors. As geometrical figures are deformed, some of the relations between its parts will change, while others will remain invariant. In some cases, by proving invariance of the relations between velocities of endpoints we will be able to prove the invariance of the relations of segments of the figure.

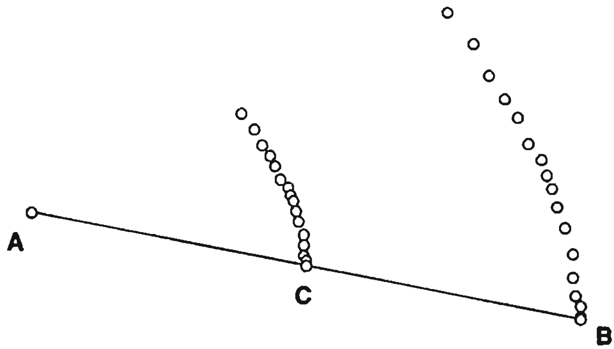

Figure 1. Relations between segments, and velocities of points.

Of course it is better if one can deform the figures oneself, rather than just look at a static figure. Using the web component of the dynamic geometric software [6], interactive figures have been posted on the web [3] so that readers have the opportunity to see the figures as they are changed. The number of the interactive figure corresponds to the figure with the same number in this paper.

RELATIONS BETWEEN VECTORS AND VELOCITIES OF ENDPOINTS

We will assume familiarity with the basic results of vector algebra and calculus. Let $\mathbf{r} = \mathbf{r}(t)$ be a vector function, with pole O and endpoint M. If at a time t_0 the vector is equal to $\mathbf{r}_0$, and at t_1 it is equal to $\mathbf{r}_1$, then the vector $\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$ will be the change of $\mathbf{r}$ in the time interval $\Delta t = t_1 - t_0$. The average velocity of the endpoint M for that time interval is given by $\Delta\mathbf{r}/\Delta t$. The instantaneous velocity of the endpoint M of the vector $\mathbf{r}$ at time t_0 is given by $v = \lim_{\Delta t \to 0} \Delta\mathbf{r}/\Delta t$.

The results about vectors and the velocities of their endpoints presented in this section are very similar to the corresponding results in calculus for functions and their derivatives. In calculus, if a relation between function holds, then the same relation will hold for their derivatives. For example, if $g(x) = 2f(x)$, then $g'(x) = 2f'(x)$. On the other hand, if a relation between derivatives holds, then the same relation holds between the functions, except that we need to add a constant. For example, if $g'(x) = 2f'(x)$, then $g(x) = 2f(x) + k$, where k is a constant.

In the same way, if we know that a relation between two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is invariant (as in Figure 1 where AB is twice as big as AC), the same relation between the velocities of their endpoints will hold (the velocity of B is twice as big as the velocity of C). On the other hand, for another pair of vectors DB and DC, if we know that the velocity of the endpoint B is always twice as big as the velocity of endpoint C, (see Figure 2), we will be able to conclude that DB is twice as big as DC, except that we need to add a constant vector $\mathbf{k}$ (in this case a vector parallel to AD).

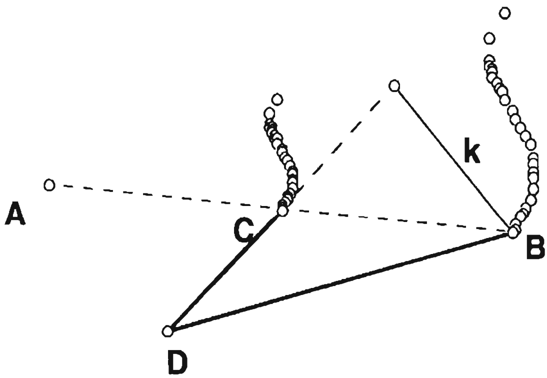

Figure 2. Velocities of endpoints and relation between segments.

The next two theorems state precisely how information about the vectors can give us information about the velocities of the endpoints and vice versa. The proofs of the results in this section can be found in [8], or any multivariable calculus book. See [2] for more illustrations of the theorems (as well as more examples of the method described here). Let $\mathbf{r}_1$, $\mathbf{r}_2$ be vectors. The respective velocities of the endpoints will be $\mathbf{v}_1$ and $\mathbf{v}_2$. Let $R_\alpha \mathbf{a}$ be the rotation of vector $\mathbf{a}$ by an angle of α degrees.

THEOREM 1. Let m be a constant number and α a constant angle. If the endpoints of vectors $\mathbf{r}_1$, $\mathbf{r}_2$ change in such a way that for all time $\mathbf{r}_1 = mR_\alpha \mathbf{r}_2$, then the velocities of their endpoints are related for all time by the equation $\mathbf{v}_1 = mR_\alpha \mathbf{v}_2$.

THEOREM 2. If the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the endpoints of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are related for all time by the equation $\mathbf{v}_1 = mR_\alpha \mathbf{v}_2$ where m is a constant number and α is a constant angle, then the vectors are related for all time by the equation $\mathbf{r}_1 = mR_\alpha \mathbf{r}_2 + \mathbf{k}$ where $\mathbf{k}$ is a constant vector.

In the following sections we show the value of this view and how these two theorems are used to obtain results in geometry.

EQUILATERAL TRIANGLE ON THREE PARALLEL LINES

PROBLEM. Given three parallel lines, construct an equilateral triangle so that one vertex of the triangle is on each of the three lines.

SOLUTION. Drop part of the condition. Construct an equilateral triangle so that two vertices G and H are on the lower parallel lines; let F be the third vertex of this equilateral triangle (Figure 3).

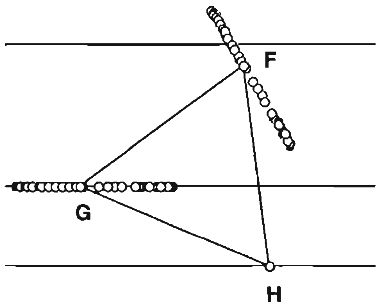

Figure 3. Equilateral triangle on parallel lines.

With H fixed, as point G is dragged along the parallel line and the rest of the figure moving accordingly so that it remains an equilateral triangle, HF always forms a 60° angle with HG, and they are the same length. Therefore the velocity of F will be the same magnitude as that of G but rotated 60° (clockwise in this case). That is, F will move also on a straight line, at 60° with the parallel lines. The desired vertex on the third line will be at the intersection of a line through F making an angle of 60° with the parallel lines and the third parallel.

This example illustrates the use of Theorem 1; knowing a relation about the segments, we know the same relation holds for the velocities of the endpoints. In each of the next examples, both theorems are used. There are three steps involved. First, use the given conditions of the problem to ascertain relations between segments of the figures. Then use Theorem 1 to infer the same relations between the velocities of the corresponding endpoints. Then use Theorem 2 to infer the same relation for another pair of segments with the same endpoints. Finally, verify that the constant vector $\mathbf{k}$ of Theorem 2 is zero, by using a special case where it is clearly true that $\mathbf{k} = 0$. The general case follows by dragging one or more vertices of the special case figure. In all examples, the proofs stand on their own and do not depend on what we see as we interact with the dynamic figures on the screen. However, the kinesthetic, dynamic use of the figures helps students to have a better and more intuitive understanding of the relations between segments and relations between velocities of endpoints.

EQUILATERAL TRIANGLES ON PARALLELOGRAM

Let $ABDC$ be a parallelogram. ACF and CDJ are equilateral triangles on adjacent sides of the parallelogram (see Figure 4). Then, BJF is an equilateral triangle.

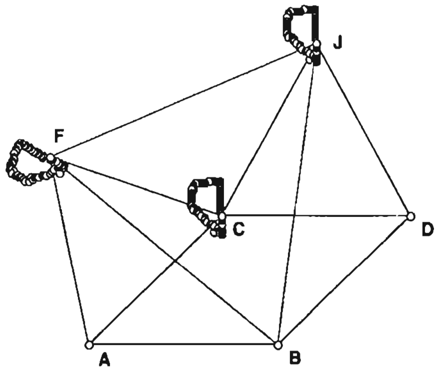

Figure 4. Equilateral triangles on parallelogram.

PROOF. D C. Points D and J will have the same velocity as C. Because AC and AF are part of an equilateral triangle, the velocity of F is the same as the velocity of C, rotated 60° (Theorem 1). Therefore the velocity of F is the same as the velocity of J, rotated 60°. Therefore BF is equal to BJ rotated 60°, plus a constant vector $\mathbf{k}$ (Theorem 2). To see that $\mathbf{k} = 0$, let C coincide with C. BF will be equal to BJ. Use the given condition that AF and AC are legs of an equilateral triangle.

ASYMMETRIC PROPELLER

Three congruent equilateral triangles share a vertex (see Figure 5). Then the midpoints of the segments connecting the vertices of the triangles form also an equilateral triangle LMN [4].

PROOF. Drag D (it will move around the circle so that the condition is satisfied). Because C and D are vertices of an equilateral triangle, the velocity of C will be the same magnitude as the velocity of D and will be rotated -60°. Because M is the midpoint of segment FD, with F fixed, the velocity of M is 1/2 the velocity of D. In the same way, the velocity of L is 1/2 the velocity of C. Therefore the velocity of M is equal to the velocity of L rotated -60°. Therefore $NM = \text{rot}\,60NL + \mathbf{k}$. To see that $\mathbf{k} = 0$, start with the three congruent equilateral triangles equally spaced in the circle. In that particular case, because of the 120° symmetry of the figure, it is clear that NML is indeed an equilateral triangle. The result is true even if the original equilateral triangles are not the same size (see Figure 6). Readers can easily adapt the previous argument.

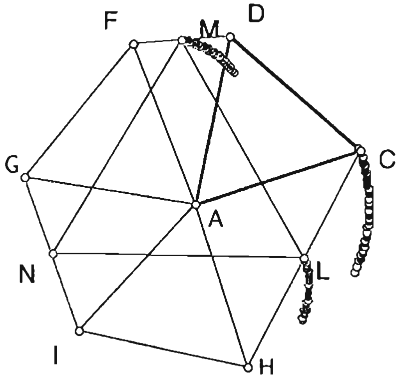

Figure 5. Asymmetric propeller.

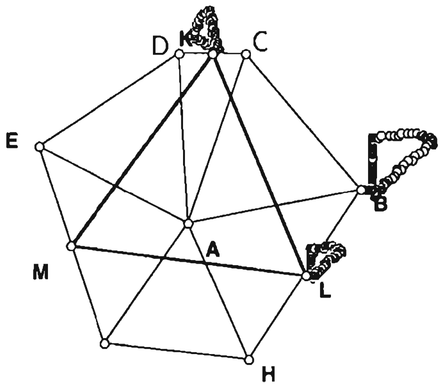

Figure 6. Propellers not the same size.

Furthermore, the result can be generalized for similar triangles of any shape. Let three triangles that are similar to each other (bold triangles in Figure 7) share the appropriate vertices at B as illustrated. Then the resulting triangle KLM formed by the midpoints of segments joining corresponding vertices of the similar triangles will be similar to the original triangles.

PROOF. Drag D. Velocity of L is 1/2 velocity of D. Velocity of K is 1/2 velocity of G. Because triangle BGD remains similar, velocity of G is equal to $BG/BD \operatorname{rot}(DBG)$ velocity of D. Therefore velocity of K = $BG/BD \operatorname{rot}(DBG)$ velocity of L. Therefore

$$MK = BG/BD \operatorname{rot}(DBG) ML + \mathbf{k}.$$

To see that $\mathbf{k} = 0$ choose original similar triangles congruent with corresponding sides parallel. Therefore MKL is similar to original triangle.

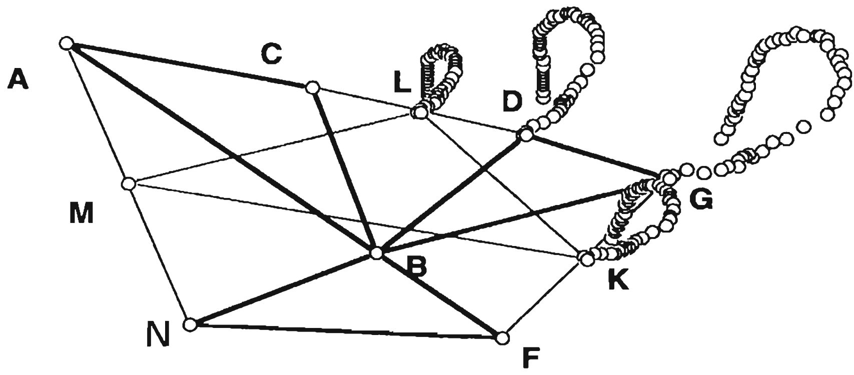

Figure 7. Propeller similar triangles.

VECTORS IN A COORDINATE SYSTEM

For the following two examples, O will be the origin located at an arbitrary place on the plane. We identify the point P with the vectorOP. We will denote the sum of two vectors $OA + OB$ simply as $A + B$. We will also use the principle that if the velocities of the endpoints are in opposite directions, and the magnitude of one is n times bigger than the other, then the point that divides the segment into two segments with a ratio 1 to n is fixed.

Centroid of a triangle. In a triangle with vertices A, B, C the point $(A + B + C)/3$ is fixed (see Figure 9). That is, it does not depend on the location of the origin, but only on the position of the three vertices of the triangle.

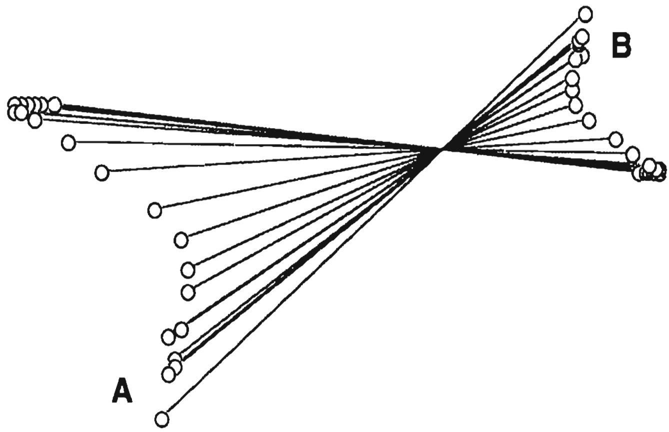

Figure 8. Fixed point.

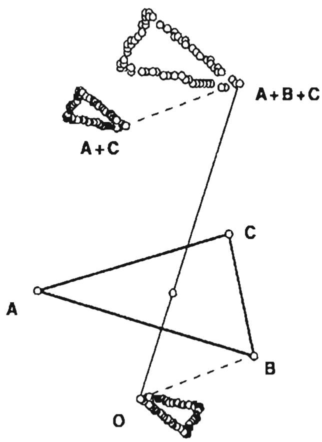

Figure 9. $(A+B+C)/3$.

PROOF. Drag O. The velocity of $A+C$ is of the same magnitude as the velocity of O, but with opposite direction, because $A+C$ is the reflection of O around the midpoint of AC. The velocity of $A+B+C$ is equal to the velocity of $A+C$ plus the opposite of the velocity of O. Therefore, the magnitude of the velocity of $A+B+C$ is twice the magnitude of the velocity of O, but in opposite direction. Therefore the point at 1/3 the distance between O and $A+B+C$ is fixed. (By the way, this is the centroid of the triangle.)

A center for a quadrilateral. In a quadrilateral $ABCD$, the point $(A + B + C + D)/4$ does not depend on the position of O (Figure 10).

PROOF. Drag O. The magnitude of the velocity of $A+B+C+D$ is the sum of the velocities of $A+D$, $B+C$, and the opposite of the velocity of O. Thus the velocity of $A+B+C+D$ has three times the magnitude of the velocity of O, and has opposite direction. Therefore the point $(A + B + C + D)/4$ remains fixed. (This point is also the point of intersection of the medians of the quadrilateral.)

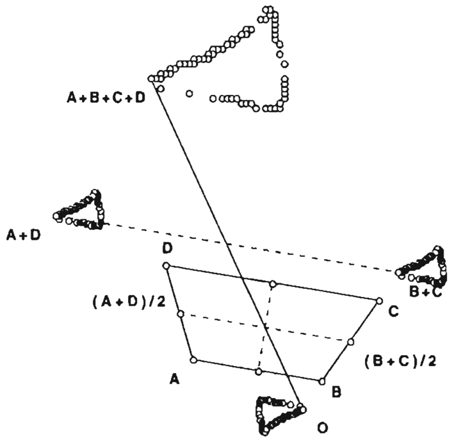

Figure 10. $(A + B + C)/4$.

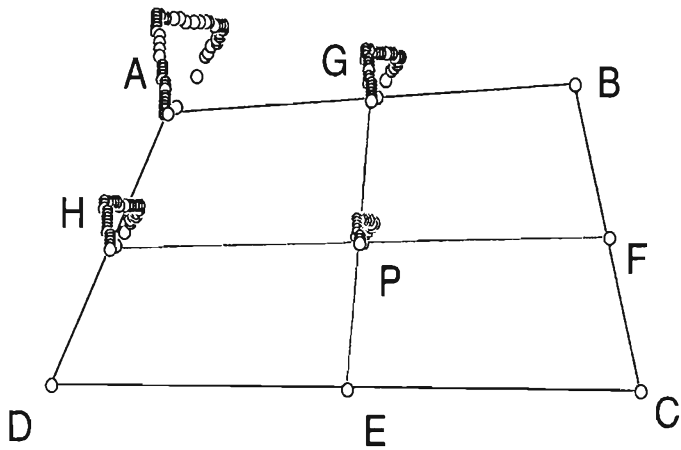

Figure 11. Joining the midpoints of opposite sides.

CENTROIDS OF THE FOUR TRIANGLES OF A QUADRILATERAL

In the following example we will consider a quadrilateral and the centroids of the four triangles associated with it. We will prove first a useful lemma.

LEMMA. Let A move with velocity $\mathbf{v}$, then the velocity of the point of intersection of the segments connecting the midpoints of opposite sides is $1/4\mathbf{v}$. (This point of intersection is also the midpoint of each of the two segments).

PROOF. Drag A. The velocity of G has the same direction as the velocity of A, but half the magnitude. The velocity of the midpoint of segment GE has the same direction as the velocity of G, but one half the magnitude, so that its velocity is $1/4$ that of A.

Remember that two figures are homothetic if they are similar and have a center of projection. We are now ready for the main result in this section.

Let ABCD be an arbitrary quadrilateral. Let $GLIM$ be the quadrilateral formed by the centroids of the four triangles BCD, ABC, ABD, and ACD (Figure 12). The two quadrilaterals are homothetic (ratio $-1/3$) and the center of homothecy is the intersection of the lines connecting midpoints of opposite sides of the original quadrilateral.

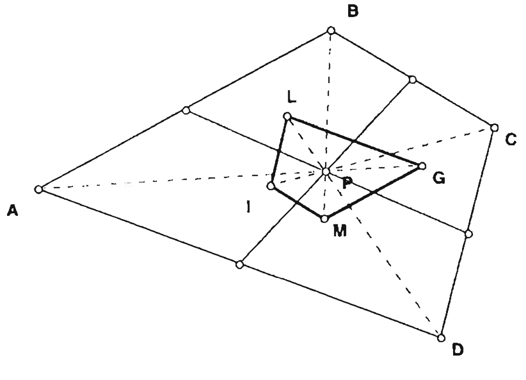

Figure 12. Homothetic quadrilaterals.

PROOF. Drag B. Let P the point that divides the segment MB in the ratio 1 to 3. The velocity of P is $1/4$ the velocity of B. The velocity of the intersection of the lines joining the midpoints of opposite sides of the quadrilateral is also $1/4$ the velocity of B. To show that they are in fact the same point start with a quadrilateral where this is clearly the case (for example a parallelogram). The general case can be obtained by dragging one or more vertices.

PARHEXAGON

On an arbitrary hexagon construct six centroids of the triangles formed by consecutive vertices such as ABC (Figure 13). The six centroids form a hexagon with three pairs of opposite sides that are congruent and parallel sides. This is called a parhexagon [7].

PROOF. Drag F. The velocities of centroids G, L, and K, are 1/3 the velocity of F, in the same direction. Therefore segments GL moves parallel to itself. (LK also moves parallel to itself.) Because the velocity of G is the same as the velocity of K then segment $HG = RK + \mathbf{k}$ (Theorem 2). To see that the segments are parallel to their opposite sides, and that they are congruent, that is, $\mathbf{k} = 0$, start with a hexagon where this is clearly true (for example, a regular hexagon).

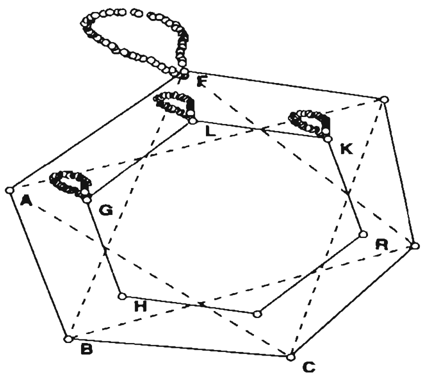

Figure 13. Parhexagon.

CONCLUSION

The results presented here can, of course, be proved using other strategies and tools. The reader may want to contrast the proofs obtained with the kinematic method with other elegant solutions [1]. As teachers of mathematics, we want to encourage students to be able find alternative ways to solve problems, and establish connections between fields such as calculus and geometry. Using the method presented here, students can use velocity to demonstrate invariance of relationships, rather than arguments of static position, length, or measurement.

Students found activities using the kinematic method "intriguing and challenging." It provided the opportunity to discover mathematics in an unfamiliar context. The dynamic geometry program played an important role to make the kinematic method easier to grasp. As one student said, "the trace at the end of the vectors helped compare and contrast two different vectors by their velocities." Another wrote "the greatest asset of both these things, the Kinematic Method and Geometer's Sketchpad is the ability to use them together to get a complete visual picture of the velocities." The interactive figures were also important. As one student phrased it: "By interacting with the figures the learning was made active in that I could make my own conjectures before I read what the answer was.... It is amazing how helpful the sketchpad can be when exploring new concepts, like velocity, and in proving the difference in velocity for different given shapes, points, lines, and angles." Several students made reference to the fact that "the study of vectors and motion has always been in a static environment" and that a visual representation like this could help students in beginning physics courses.

Of course, the method can be applied to other examples. Students may want to try to provide their own proofs for the cases illustrated in the interactive Figures 6a, 7a, and 7b [3].

The kinematic method is not new and students can use it without a dynamic geometry program. However, the possibility of varying the data, of experimenting, and interacting can help student to better comprehend the relations between vectors and their velocities that are used in the proofs.

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BIOGRAPHICAL SKETCH

Alfinio Flores has degrees in mathematics from UNAM in Mexico, and a PhD in Mathematics Education from The Ohio State University. Currently he is Professor of Mathematics Education at Arizona State University. One of his interests is to explore ways in which technology can help make mathematics more interesting and engaging for a broader segment of the population.