Two generalizations of the Napoleon theorem

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Abstract

I give two generalizations of the Napoleon theorem. The first generalization associated with a hexagon. The second generalzation associated with the Kiepert hyperbola.

Theorem 1. *Let $ABCDEF$ be a hexagon, constructed three equilaterals AGB, CHD, EIF all externally or internally (as in the figure 1). Let A_1, B_1, C_1 be then the centroid of FGC, BHE, DIA respectively. Let A_2, B_2, C_2 be the centroid of EGD, AHF, CIB respectively. Then $A_1B_1C_1$, and $A_2B_2C_2$ form an equilateral triangle.*

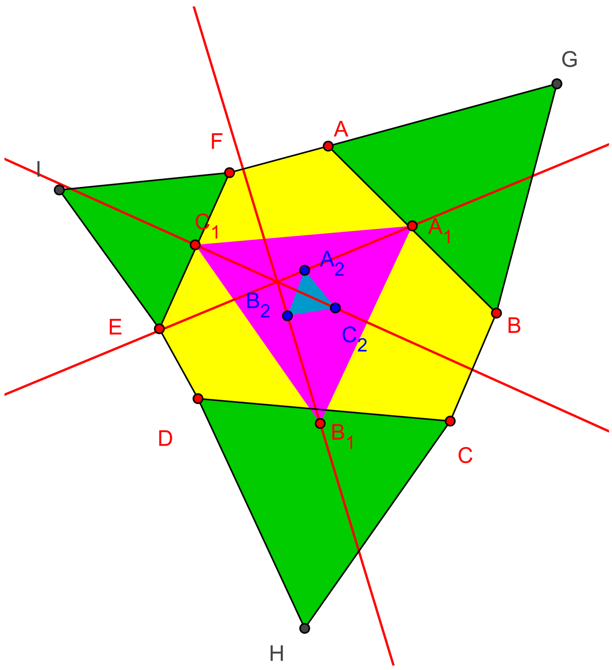

Figure 1

The theorem 1 publish by the author in Cut The Knot web site, you can see the proof of theorem 1 by complex number in [1].

When we put $A = F, B = C, D = E$ theorem 1 is the Napoleon theorem.

Proposition 2. *Two triangles $A_1B_1C_1$, and $A_2B_2C_2$ are perpective.*

Theorem 3. *Let ABC be a triangle, F be the first (or second) Fermat point, let K be the point on the Kiepert hyperbola. Let P be the point on line FK. The line through P and perpendicular to BC meet AK at A_0. Define A_0, B_0, C_0 cyclically. Show that $A_0B_0C_0$ is an equilateral triangle. This triangle homothety to the outer(or inner) Napoleon triangle.*

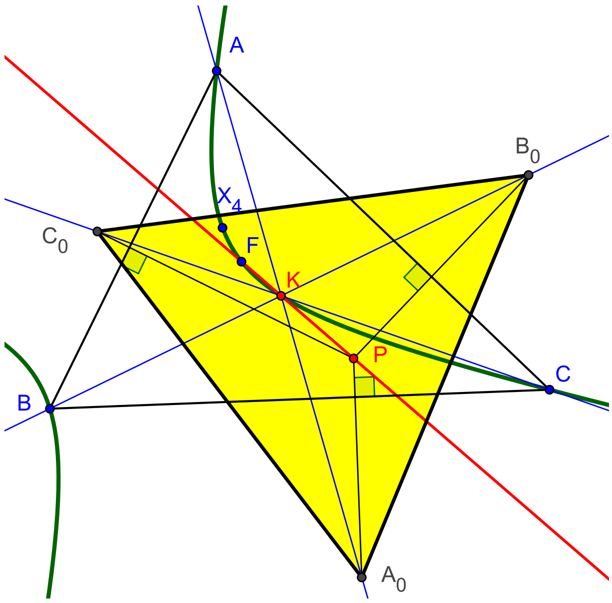

Figure 2

When ($F = X(13)$ and $K = X(17)$ and $P = X(3)$) or ($F = X(14)$ and $K = X(18)$ and $P = X(3)$) the triangle $A_0B_0C_0$ is outer or inner Napoleon triangle respectively.

The theorem 3 publish me in [2]. A proof of theorem 3 by **Telv Cohl** as follows:

Lemma 4. *(USA TST 2006, Problem 6) Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.*

Let O' be the circumcenter of $\triangle ACQ$. Let M, N be the midpoint of CQ, CR, respectively .

Easy to see $R \in (O')$.

Since O', M, N, C are concyclic , so we get $\angle AO'O = \angle QCP$ . ... (1) Since $\angle RO'O = \angle BQC, \angle O'OR = \angle CBQ$, so we get $\triangle ORO' \sim \triangle BCQ$, hence $\frac{O'A}{CQ} = \frac{O'R}{CQ} = \frac{O'O}{QB} = \frac{O'O}{CP}$ . ... (2)

From (1) and (2) we get $\triangle AOO' \sim \triangle QPC$, so from $OO' \perp PC$ and $AO' \perp QC \Longrightarrow AO \perp QP$.

Lemma 5. *Let D be a point out of $\triangle ABC$ satisfy $\angle DBC = \angle DCB = \theta$. Let E be a point out of $\triangle ABC$ satisfy $\angle EAC = \angle ECA = 90° - \theta$. Let F be a point out of $\triangle ABC$ satisfy $\angle FAB = \angle FBA = 90° - \theta$. Then $AD \perp EF$.*

Let $B' \in AF, C' \in AE$ satisfy $AB = AB', AC = AC'$ and $T = BC' \cap CB'$.

Easy to see $\triangle ABB' \cup F \sim \triangle ACC' \cup E \Longrightarrow EF \parallel B'C'$.

From $\triangle AB'C \sim \triangle ABC' \Longrightarrow \angle BTC = 180° - (90° - \theta) = 90° + \theta$, so combine with $\angle DBC = \angle DCB = \theta$ we get D is the circumcenter of $\triangle BTC$, hence from lemma 1, we get $AD \perp B'C'$. i.e. $AD \perp EF$

From the lemma we get the following property about Kiepert triangle : The pedal triangle of the isogonal conjugate of $K_{90-\phi}$ WRT $\triangle ABC$ and the Kiepert triangle with angle ϕ are homothetic . (Moreover, the homothety center of these two triangles is the Symmedian point of $\triangle ABC$!) (1)

Let H_b, H_c be the orthocenter of $\triangle FCA, \triangle FAB$, respectively . ($H_b, H_c$ also lie on the Kiepert hyperbola of $\triangle ABC$)

Easy to see all $\triangle A_0B_0C_0$ are homothetic with center K , so it is suffices to prove the case when P coincide with F .

From Pascal theorem (for $CKBH_cFH_b$) we get $AF \perp B_0C_0$. Similarly, we can prove $BF \perp C_0A_0$ and $CF \perp A_0B_0$, so $\triangle A_0B_0C_0$ and the pedal triangle of the isogonal conjugate of F WRT $\triangle ABC$ are homothetic , hence from (1) we get $\triangle A_0B_0C_0$ and the outer (or inner) Napoleon triangle are homothetic .

You can see Telv Cohl's proof in [3][4][5].

References

[1] A. Bogomolny, A Final Chapter of the Asymmetric Propeller Story, Junly 2013. Available at http://www.cut-the-knot.org/m/Geometry/FinalAsymmetricPropeller.shtml

[2] O. T. Dao, Advanced Plane Geometry, message 2261, January 24, 2015.

[3] http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=622242

[4] http://www.artofproblemsolving.com/Forum/viewtopic.php?f=48&t=621954

[5] http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=148830

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