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RESEARCH EVIDENCE ON HIERARCHICAL THINKING, TEACHING STRATEGIES AND THE VAN HIELE THEORY: SOME CRITICAL COMMENTS

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ABSTRACT

This paper reviews two recent studies which suggest a radical rethink of the place of hierarchical classification in the Van Hiele theory. Results from both studies indicate that hierarchical class inclusion may develop independently from deductive thinking. Theoretical as well as empirical evidence suggest that the ordering of the various stages of development in the Van Hiele model is not necessarily independent of the teaching strategy used.

INTRODUCTION*

Plane geometry in the Durell (1939) style forms an important component of the high school mathematics curriculum in South Africa. Informal geometry is taught from Grade 4 to Grade 8, while in Grade 9 the curriculum prescribes an axiomatized version of elementary plane geometry. This emphasis on a formalized deductive system in Grade 9 seems to have caused many learning problems.

The author is presently project leader of the RUMEUS research project on geometry, a continuation of the USEME teaching experiment of 1977/78 (Human et al, 1984). Further studies have focussed on developing testing material for certain mathematical processes in geometry (Joubert, 1980), as well as using groupwork for the teaching of the properties of quadrilaterals and their classification (De Vries, 1980).

This article discusses one of the theoretical starting points, namely the Van Hiele theory, which has influenced and partly guided our research at RUMEUS into students' problems in learning formal geometry. In turn, our teaching experiments and research findings have strongly influenced our understanding of the Van Hiele model and even led to some refinement and elaboration. In this article the results of two recent research studies by Malan (1986) and Njisane (1986) will be reviewed in relation to the Van Hiele model.

THEORETICAL FRAMEWORK

THE VAN HIELE LEVELS

The most important aspect of the Van Hiele theory is the distinction of five levels in the mastery of geometry, and the hypothesis that they form a learning hierarchy. Thus, someone cannot be at a specific level without having passed through the preceding levels. Throughout this paper the numbering system of Usiskin (1982) and Senk (1983 & 1984) will be used for the levels. The general characteristics of each level is given below, as adapted from Usiskin (1982), Senk (1983) and Hoffer (1983):

* The research projects reported in this paper were supported by the South African Human Science Research Council (HSRC). Any opinions, findings, conclusions or recommendations expressed are those of the author and the researchers involved and do not necessarily reflect the views of the HSRC.

Level 1: Recognition. Students recognize figures by their global appearance. They recognize triangles, squares, parallelograms, and so forth, but they do not explicitly identify properties of these figures.

Level 2: Analysis. Students analyse properties of figures and learn the appropriate technical terminology for describing them, but they do not explicitly interrelate figures or properties of figures.

Level 3: Ordering. Students logically order properties of figures by short chains of deduction and understand interrelationships between figures (e.g. class inclusions).

Level 4: Deduction. Students develop longer sequences of statements to deduce one statement from another, and also understand the significance of deduction, the role of axioms, theorems and proof.

Level 5: Rigor. Students analyse various deductive systems with a high degree of rigor, while understanding such properties of a deductive system as consistency, independence and completeness of axioms.

For more information on the levels and the nature of the theory, consult Wirszup (1976), Mayberry (1981, 83), Fuys (1986) and a recent study by Burger & Shaughnessy (1986) in this journal.

HIERARCHICAL CLASS INCLUSION

Although there seems to be a consensus amongst the above-mentioned American theorists and researchers that hierarchical class inclusion (e.g. a square is a rectangle) occurs at level 3 (Ordering), there seems to be some confusion in the Van Hiele literature itself. For instance, in **Begrip en Inzicht** Pierre van Hiele argues as follows that class inclusion can already occur at level 2 (their Level 1):

> "The development of a network of relations results in a rhombus becoming a symbol for a large set of properties. The relationship of the rhombus to other figures is now determined by this collection of properties. Students, who have progressed to this level, will answer the question of what a rhombus is by saying: 'A rhombus is a quadrilateral with four equal sides, with opposite angles equal and with perpendicular bisecting diagonals which also bisect the angles.' On the grounds of this, a square now becomes a rhombus."[2]

[2] Freely translated from the original Dutch (Van Hiele, 1973:93).

This same point is made by Dina van Hiele in Fuys et al (1984: 222) when writing:

> "... at level zero, a square is not perceived as a rhombus; at the first level of thinking, it is self-evident that a square is a rhombus."

However, Pierre van Hiele in Fuys et al (1984: 245) seems to contradict himself and his wife when writing with reference to Level 2 (their first level):

> "But at this level ... a square is not necessarily identified as being a rectangle."

Consequently, one of the objectives underlying our research has been to try and clarify the level at which hierarchical classification is supposed to occur: does it precede the informal deductive thinking of Level 3, or coincide with it?

DEDUCTIVE REASONING AND PROOF

According to the Van Hiele theory, deductive reasoning first occurs on Level 3 when the network of logical relationships/ implications between properties is established, while the meaning of formal deduction and proof is only understood at the next level. Students who are on Levels 1 and 2 with regard to a specific topic will not understand instruction aimed at the activities and meanings of the higher levels. For instance, when a teacher provides a deductive proof that the diagonals of a rectangle are equal, the meaning of the proof lies in expliciting the logical relationships between the properties, not in establishing the validity of the properties themselves. A student at Level 1 or 2, who does not possess this network of logical implications, experiences such a proof as an attempt at the verification of the result. However, since he (or she) does not doubt the validity of his (or her) empirical observations, he (or she) experiences such a proof as meaningless: "proving the obvious". The Van Hiele theory therefore strongly criticizes any form of geometry teaching wherein deductive reasoning and proof plays a major role, if students cannot yet see the meaning of it in terms of logical systematization (Van Hiele ,1973: 97) .

However, since proof in mathematics has other meanings than systematization, we believe that proof may be meaningful to students at levels lower than Level 4. In accordance with Bell (1976: 24) we distinguish the following three meanings or functions of proof in mathematics:

- verification (concerned with the truth of a proposition).
- explanation (conveys insight into why it is true).
- systematization (the organization of results into a deductive system).

Since the everyday usage of the term "proof" conveys with it a meaning of "convince", "checking", "making sure", "removing all doubts", we believe that students' first encounter with proof should preferably be within the context of the verification (or explanation) of some startling or surprising results and not in the context of systematization. In the USEME teaching experiment in 1977-1978 in 10 schools, students were successfully introduced to situations they had to "explain" or "verify" relatively early (Human et al, 1984). For instance, explaining why a parallelogram was always formed by connecting the midpoints of the adjacent sides of a quadrilateral. It seems that proof in this sense may perhaps be experienced as meaningful by students even at Level 2. This interpretation of the various meanings of proof and their relationships to the Van Hiele levels was also successfully implemented in a teaching experiment with Boolean Algebra (De Villiers, 1986b).

Closely associated with children's understanding of the significance of deduction, is their perception of the nature of axioms and their role in mathematics. We believe that the level change from Level 3 to Level 4 also involves a change from a classical perspective on the nature of axioms (intuitively accepted) to a modern perspective (accepted as hypothetical starting points of a mathematical system). Freudenthal (1973: 451-461) has furthermore pointed out that the axiomatization of geometry should be carried out in stages, first "locally" and then gradually more "globally". Although local and global axiomatization respectively seem to correspond to Level's 3 and 4, we feel that this distinction does not really involve a level change, but is merely a progressive sophistication of thought.

NJISANE'S STUDY

MAIN OBJECTIVE OF THE RESEARCH

This research project (Njisane, (1986)) was aimed at finding out if different geometric thought categories (GTC's) form Guttman scales and how they correspond with the Van Hiele model.

SAMPLE

The sample consisted of 4015 high school pupils in grades 9 to 12, in May/June 1984. This was the total population of pupils taking mathematics in a radom sample of high schools of the Kwazulu Department of Education, situated in the province of Natal. The schools ranged from small, rural schools to big inner-city schools.

THE INSTRUMENT

The test consisted of 56 open-ended questions ranging from simple questions like indicating alternate angles when parallel lines were given, listing the properties of a given figure like a parallelogram, to questions requiring the interpretation of formal definitions and the construction of formal proofs. Most items dealt with traditional content like parallel lines, perpendicular lines, technical terms, isosceles triangles, congruent triangles, parallelograms, rectangles, logical inferences, significance of deduction, perspective on the difference between an axiom and a theorem, etc. Furthermore in several items, pupils were asked to give reasons for their responses providing extremely useful information on the nature of their conceptual understanding. A copy of the test and marking scheme is available on request.

PUPILS

Njisane's research was aimed at establishing a non-prejudiced description of progress in geometric thinking. This was facilitated by distinguishing a number of different geometric thought categories (GTC's) each being reflected in a number of test items. The most important categories are:

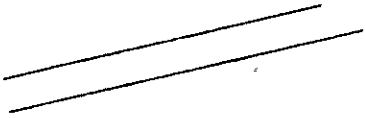
- recognition and representation of figure-types (Z) (Van Hiele 1)
- use and understanding of terminology (A) (Van Hiele 2)
- verbal description of properties of a figure type (E) (Van Hiele 2)
- hierarchical classification (B) (Van Hiele 3)
- one step deduction (C) (Van Hiele 3)
- longer deduction (D) (Van Hiele 4).

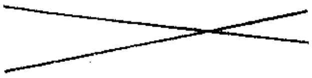
[Place figures 1 and 2 more or less here]

Examples of test items testing each of the above GTC's are given in Figures 1 and 2. Details on how students performed on the full range of GTC's are given in De Villiers and Njisane (1987). Using a 50% criterion for proficiency in a specific GTC, it was found that hierarchical classification (B) was the most difficult, closely followed by longer deduction (D) and one step deduction (C) (See Table 1). From these results it is clear that these students find hierarchical classification much more difficult than the other GTC's. It is also significant that, compared to the other GTC's, very little improvement in hierarchical classification occurred through the grades. Guttman analyses were also done on the results for all the GTC's and various subsets thereof, producing reproducibility coefficients and scalability coefficients above the required criteria of 0,9 and 0,6 respectively. This suggests a learning hierarchy among the GTC's in Table 1 from left to right.

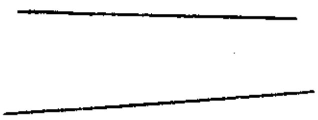
[Place table 1 more or less here]

1. Write YES if the two lines look like parallel lines.
Write NO if the two lines are not parallel.
Write I DO NOT KNOW if you do not know whether the lines are parallel or not parallel.

(a)  ....

(b)  ....

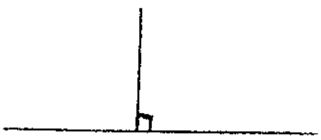
(c)  ....

(d)  ....

2. Draw a right-angled triangle

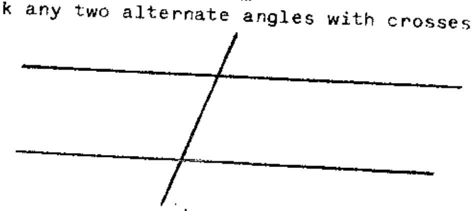
(Recognition and representation of figure-types (Z))

3. What are lines like the following called?



....

4. Mark any two alternate angles with crosses.



(Use and understanding of terminology (A))

5. Describe all the different properties of the parallelograms.

6. What is a rectangle?

(Verbal description of properties of a figure-type (E))

7. (a) How big is $\hat{3}$ in the figure given below?
(The figure is drawn accurately)

....

(b) How did you obtain your answer?

8. A line PQ meets another line AB at Q such that $A\hat{Q}P$ = B
(a) Is PQ ⊥ AB? ....
(b) Give reasons for your answer.

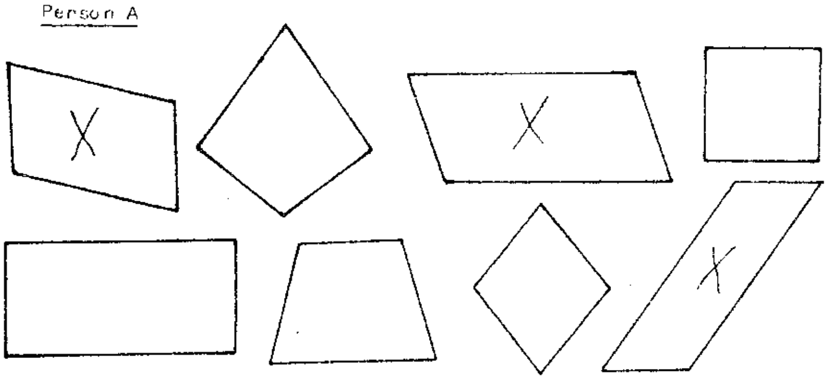
(One step deduction (C))

FIGURE 1: EXAMPLES OF TEST ITEMS TESTING GTC's

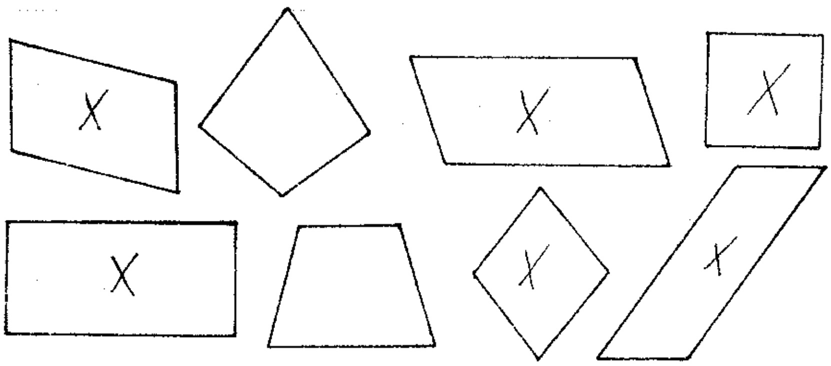
1. Two different persons were asked to indicate all the parallelograms in a given set of figures with crosses.

(a) Which person correctly indicated the parallelograms (A or B or NOBODY)? ....

Person A



Person B

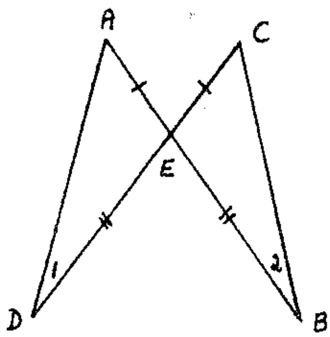


2. (a) Is every square also a rectangle?

(b) Give reasons for your answer.

(Hierarchical classification (B))

3. In the figure AB and CD are straight lines that meet at E such that AE=EC and DE=EB

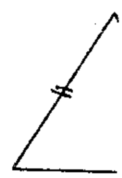


(a) What can you say about $\hat{1}$ and $\hat{2}$? ....

(b) Why do you say so?

4. (a) Are you convinced (sure) that the base angles of an isosceles triangle are always equal?

....



(b) Explain why you are sure or why you are not sur

(Longer deduction (D))

FIGURE 2: EXAMPLES OF TEST ITEMS TESTING GTC's

GRADE	Z	A	E	C	D	B	NUMBER OF STUDENTS
9	38,8%	11,6%	5,3%	2,5%	0,2%	0,5%	1192
10	58,8%	30,9%	23,5%	14,0%	2,9%	1,7%	1697
11	54,3%	68,1%	65,4%	44,3%	22,2%	5,0%	655
12	90,0%	84,0%	68,3%	63,3%	42,6%	5,1%	379

TABLE 1 STUDENT PERFORMANCES ON GTC's

From these results the following general conclusions were drawn:

(i) Although the GTC's generally support the Van Hiele model, hierarchical thinking seems neither a prerequisite for deductive thinking (a Level 2 characteristic) nor does it seem to develop in conjunction with the development of the logical implications between properties (a Level 3 characteristic).

(ii) Simpler one-step deduction may be possible at levels lower than Level 3 or 4, as evidenced by the fact that the GTC's verbal description (E) and one step deduction (C) were more or less of the same level of difficulty (See Table 1) and continually interchanged in the Guttman analyses, depending on the division points chosen (Njisane, 1986).

(iii) The fact that very little improvement over the grades occurred in hierarchical classification compared to deduction, is probably in part attributable to the fact that the study of quadrilaterals (and their classification) terminates in Grade 10, while deduction as a process is still focussed upon until Grade 12, although the content matter is then properties of circles and similarity of triangles.

MALAN'S STUDY

MAIN OBJECTIVE OF RESEARCH

The main objective of Malan's research was to investigate teaching methods for facilitating childrens' transition from partition to hierarchical classification. The sample consisted of 14 children chosen at random from schools in the vicinity of the University of Stellenbosch. All the interviews were conducted in Afrikaans, and a translated version of the questions used and interview samples are given below.

RESEARCH METHOD

Clinical interviews ranging from 1 hour to $1\frac{1}{2}$ hours were conducted with individual students. The interviews consisted of two parts. Section A consisted of a questionnaire and was aimed at determining the Van Hiele level of the student. In Section B, however, an attempt was made to lead those who were still partition classifiers to being hierarchical classifiers via the discussion of the properties of quadrilaterals. The second part was therefore a teaching experiment aimed at identifying and evaluating possible ways of leading pupils to hierarchical classification of quadrilaterals. An observer, familiar with the Van Hiele levels, was always present, and at the conclusion of each session he separately questioned each pupil. The interviews were all audio-taped and transcribed for later reference and final analysis.

QUESTIONNAIRE

To test for Level 1 Van Hiele thinking (recognition), the children were first given a sheet with sketches of various quadrilaterals and asked to name them. (The sheet was similar to one used by Burger and Shaughnessy (1986)). Although most children did well on this activity, there were four children who identified less than half the figures.

Thereafter they were given Level 2 questions (analysis) which tested their identification of properties of quadrilaterals. Some examples are given below:

"Circle all the words in brackets which convert the sentences below into true sentences:

i) All the sides of a (square, parallelogram, quadrilateral, trapezium, rhombus, kite, rectangle) are equal.

ii) If a quadrilateral ABCD is a (square, parallelogram, quadrilateral, trapezium, rhombus, kite, rectangle), then at least one pair of opposite sides are parallel.

iii) Both pairs of opposite sides of a (kite, quadrilateral, square, trapezium, rhombus, parallelogram, rectangle) are equal.

iv) The diagonals of a (square, parallelogram, quadrilateral, trapezium, rhombus, kite, rectangle) are equal.

v) The diagonals of a (kite, rectangle, square, rhombus, parallelogram, quadrilateral, trapezium) are perpendicular."

In the above question some children needed reminding that more than one answer could be correct, since some stopped at the first correct answer, moving to the next question. For Level 3 thinking children were asked to complete questions like the following:

"What type of quadrilateral is ABCD in each of the following cases?

i) AB//DC, AD//BC, $\hat{A} = \hat{C}$, $\hat{B} = \hat{D}$, AB = DC, AD = BC, AC and BD bisect each other

ABCD is a ....

ii) AB = CD, AD = BC, AC = BD, $\hat{A} = \hat{B} = \hat{C} = \hat{D} = 90°$, AB//CD, AD//BC

ABCD is a ....

iii) AD//BC, AD = BC

ABCD is a ....

iv) AB = BC = CD = AD

ABCD is a ....

v) AB//DC, $\hat{A} = 90^\circ$

ABCD is a ...."

TEACHING EXPERIMENT

This part was extremely complex and varied considerably depending on the individual reaction of each student. However, it is precisely in this adaptability that lies the power of one-to-one interview-teaching situations such as this, namely: a much more accurate probing of each student's conceptual understanding than by any other technique.

[Place figure 3 more or less here]

A broad outline of the protocol procedure which was used in this section, is given in Figure 3. Only the main part of the procedure will be described here. Firstly it was determined whether the student was a partition or hierarchical classifier by showing him/her a rectangle drawn on a sheet and asking if it was a parallelogram. If they said it was not a parallelogram, but a rectangle, they were asked to explain what a parallelogram was. If they gave a definition of properties which allowed for hierarchical inclusion, they were asked to check if rectangles also had these properties and if we could therefore say that rectangles were (special) parallelograms. On the other hand, if they gave a partition definition which excluded rectangles, either other figures were tried or it was sometimes tried to emphasize the similarities between them by using a table. Since some students seemed to give as reason for their exclusion of rectangles, "a mathematical object cannot have two names", they were shown that, for instance, in biological classifications a biological object may at the same time be a vertebrate and a mammal or a vertebrate and a reptile: or in mathematics , that a rectangle is both a rectangle and a quadrilateral. With some students a dynamic approach was tried by showing, for instance, that a parallelogram (with sides of fixed length) could be transformed into various shapes of which a rectangle is a special case as shown in Figure 4. This was unfortunately only done by drawings and not by a physical model.

[Place figure 4 more or less here]

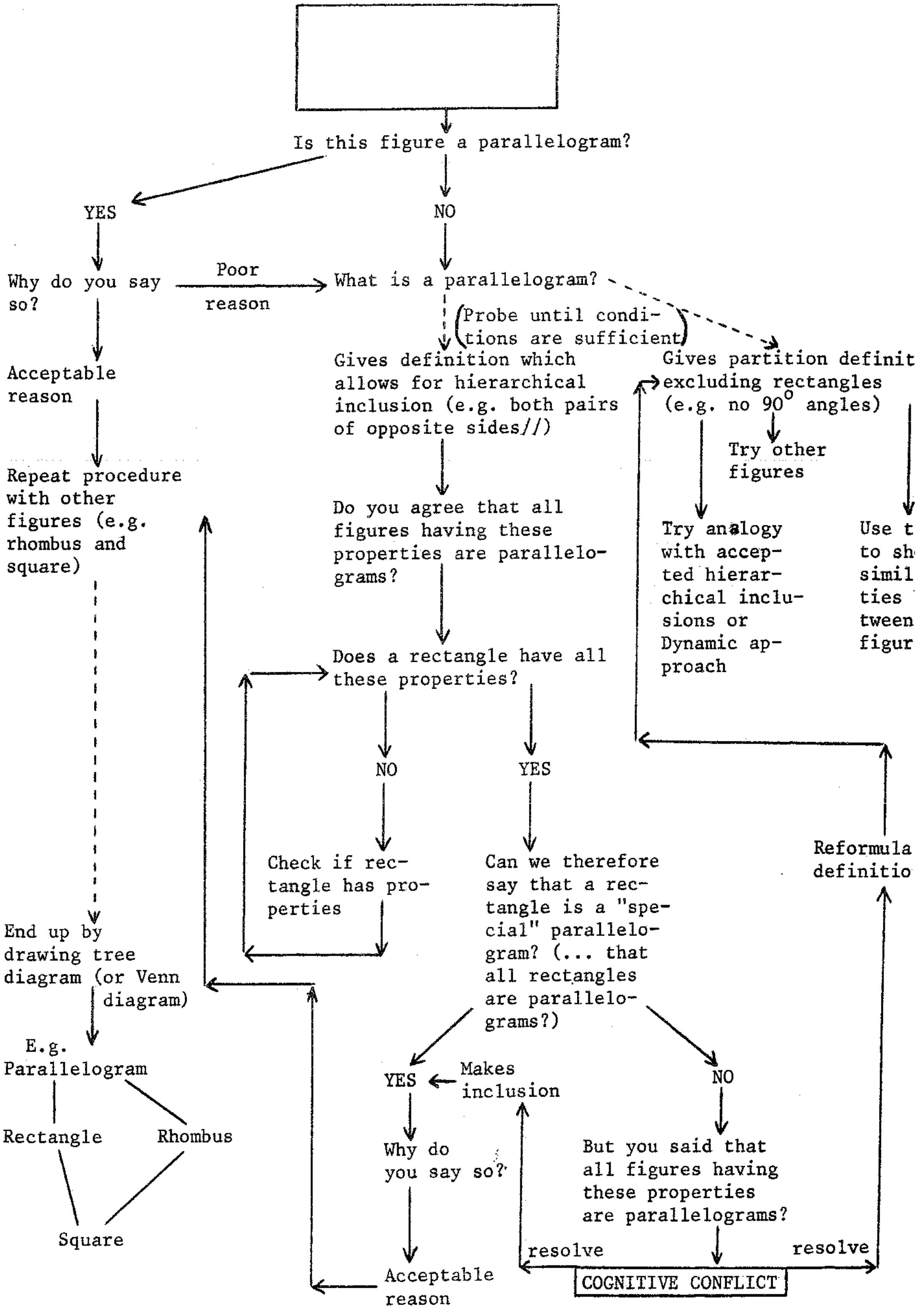


FIGURE 3. OUTLINE OF PROTOCOL PROCEDURE OF SECTION B

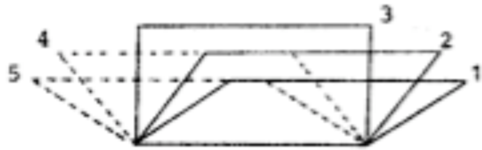


FIGURE 4. SOME TRANSFORMATIONS OF A PARALLELOGRAM WITH FIXED SIDES

Complete the following table by marking ✓ if the quadrilateral has that property and X if it does not have it.

	SQUARE	RECTANGLE	PARALLELO-GRAM	KITE	TRAPE-ZIUM
At least one pair of opposite sides parallel					
Both pairs of oppo-site sides parallel					
Both pairs of oppo-site sides equal					
All four sides equal					
Two pairs of adjacent sides equal					

TABLE 2 COMPARING THE PROPERTIES OF QUADRILATERALS

Students who responded positively to the inclusion of rectangles among the parallelograms were asked to justify their answer by giving reasons. If the reasons were acceptable, the process was repeated with other figures, and if time allowed it, they were led to draw tree diagrams (or Venn diagrams). The following are examples of some of the questions which were asked at this time: "What types of figures are both kites and parallelograms?", "Are any kites rectangles? If so, which?" and to check their thinking: "Are all rectangles squares?" and "If a figure is not a kite, can it be rhombus?"

Results

Illustrative responses

From the 14 students interviewed, a sample of 6 has been selected for reporting here. This sample was chosen to be fairly representative of the variety of responses the students exhibited. The students are Lenie, San, Carin, female 8th graders; Rudolf, a male 8th grader; and Dolf and Gawie, male 9th graders. (These are not their real names.)

Lenie

To the first question in Figure 3 she responded negatively, after which she was asked to complete a table for all the various properties of quadrilaterals. An example of a subset of this table is shown in Table 2.

[Place table 2 more or less here]

Contrary to our expectations, this table did not help her to see the similarities between the various quadrilaterals, and seemed rather to strengthen her partition thinking. At the end she gave the following definition for a parallelogram:

"A quadrilateral with opposite sides equal and parallel, opposite angles equal, diagonals of different length halving each other, but not perpendicularly"

Interviewer (I): "Must the diagonals of a parallelogram be of different length?"

Student (S): "If they are equal, then the angles are 90° and then it is a rectangle".

Rudolf

This student also responded negatively to the first question, and was then asked to describe what a parallelogram was, giving an uneconomical definition (i.e. a lot of redundant properties). A rectangle was then tested against these conditions and although he agreed that a rectangle had all the properties of a parallelogam, he resisted any inclusion, saying: "But the diagonals of a rectangle are also equal."

After testing if the rhombi had all the properties of parallelograms, he also refused any inclusion, because (he said):

> "if all four sides are of equal length, then it is no longer a parallelogram, but a rhombus."

After the completion of Table 2 for squares and rhombi the following discussion occurred:

I: "Between the properties of these two figures there are many similarities. The differences that are there, is that the diagonals of a square are equal and that all its angles are 90°, while that is not the case with the rhombi. Is it sufficient to say that a rhombus is a quadrilateral with all four sides equal?"

S: "No, then it can also be a square."

I: "But it doesn't have to be a square, it may also be skew figure."

(Draws quadrilaterals with four equal sides: one with 90° -angles, and the other one not.)

"Is a square then not a type of rhombus; a special kind of rhombus?"

S: "No"

I: "Can't one say that a square is a rhombus with 90° degree angles?"

S: "No, a square is a square."

I: "So, a rhombus must be a skew figure?"

S: "Yes."

San

The first figure in Figure 3 to her was a rectangle, but not a parallelogram. Her definition for a parallelogram was:

> "Two pairs of opposite sides parallel, but not all equal, because then it is a rhombus"

clearly disallowing the hierarchical inclusion of the rhombi. However, her "definition" for a rectangle was: "very similar to a parallelogram, but with the lines not skew" showing some potential for hierarchical inclusion. The interviewer therefore continued:

I: "Can I therefore say that a rectangle is a parallelogram with angles equal to 90°?"

S: "Yes"

I: "But then the above figure is a parallelogram?"

S: "Yes" (confidently)

Although her initial definition for a rhombus excludes squares (no right angles allowed), she eventually defined a square as a special rhombus with right angles. Comparing the above definitions for a square and a rectangle respectively, she immediately said that a square could also be viewed as a special kind of rectangle, with an extra property namely, equal sides.

However, after completing Table 2 she was again confused when asked whether a square was a rectangle, saying that that doesn't make sense, since "then there would be no difference between them". After checking that a square has all the properties of a rectangle (but not vice versa), she agreed that although a square is a rectangle, a rectangle is not a square.

After a comparison of the properties of a parallelogram and rectangle, she made the appropriate hierarchical inclusion. Asking her to visually compare a rhombus and a kite (without analysing their respective properties), was now sufficient for her to conclude that a rhombus is a kite.

Carin

Carin immediately accepted the given rectangle as a parallelogram basing her decision on the economical definition that a parallelogram was a quadrilateral with both pairs of opposite sides equal. This definition she also applied to include the squares and rhombi. However, she refused to include the squares with the rectangles, even after comparing their properties. It was only after repeated comparisons between the parallelogram-rectangle relation (which she accepted hierarchically) and the rectangle-square relation, that she committed herself to accepting a square as a rectangle.

This was followed by the construction of a tree diagram. The following conversation then took place with the placement of the rhombi:

I: "Is a rhombus a quadrilateral?"

S: "Yes"

I: "Is a rhombus a parallelogram?"

S: "Yes"

I: "Is a rhombus a rectangle?"

S: "No"

I: "Is a rhombus a square?"

S: "No"

I: "Is a square a rhombus?"

S: "Yes"

I: "Where must the rhombi therefore be placed in the diagram?"

S: "Beneath the parallelograms, but it has nothing to do with the rectangles."

She eventually also correctly placed the trapeziums and kites in the diagram, and could answer questions in regard to the intersections of the various types of quadrilaterals.

Dolf

This student also initially refused to make class inclusions as shown in the following conversation which occurred after a rectangle had been tested against the properties of a parallelogram:

I: "Is any figure with these properties a parallelogram?"

S: "I don't think so ... This is rather a difficult one ... I should say yes."

I: "So this figure (pointing to a rectangle) is a parallelogram?"

S: "No, because we did not say that these two angles must be equal. Something else must be included to prevent us calling a rectangle a parallelogram."

I: "A little while ago we said that this figure (pointing to a rectangle) was both a quadrilateral and a rectangle."

S: "Yes, because a rectangle is a quadrilateral with special properties."

I: "But can't I then say that a rectangle is a parallelogram with special properties?"

S: "But why do we have two names if we can say a rectangle is a parallelogram?"

However, after the properties of a parallelogram was again discussed, a sudden resolution occurred:

I: "With the rectangle we found all these properties, as well as the property that the angles are all 90°. Can we therefore say that a rectangle is a parallelogram, a special type of parallelogram?"

S: "Yes" (without hesitation).

The properties of rectangles and squares where then written down and compared.

I: "Does a square have all the properties of a rectangle?"

S: "Yes"

I: "So it is a rectangle?"

S: "No, it is a square."

I: "A special rectangle?"

S: "Yes, a rectangle with all its sides equal."

I: "Is a square also a parallelogram?"

S: "Yes (surprised) ..., because it has all the properties of a parallelogram."

He was finally led to draw up a hierarchical schema for the various quadrilaterals studied in the curriculum.

Gawie

In Section A of the interview he could only recognize squares, rectangles and kites. The names of the other quadrilaterals were unknown to him. He also virtually knew none of the properties of the various quadrilaterals.

A completely different approach was then followed with Gawie.

The following concepts were first clarified: the size of an angle, the length of a side, parallelness, and the meaning of the words "at least". Certain conditions were then placed on the concept quadrilateral by selecting certain properties. Gawie was then shown a number of sketches of quadrilaterals, and he had to decide which complied with the chosen conditions. Afterwards the name of the set of quadrilaterals complying with the restriction was given. An example is given in Figure 5.

[Place figure 5 more or less here]

He was also asked to identify those quadrilaterals with both pairs of opposite sides parallel and those with both pairs of opposite sides equal and all four sides equal. Gawie was then asked to explain what a rectangle was. He did this as follows:

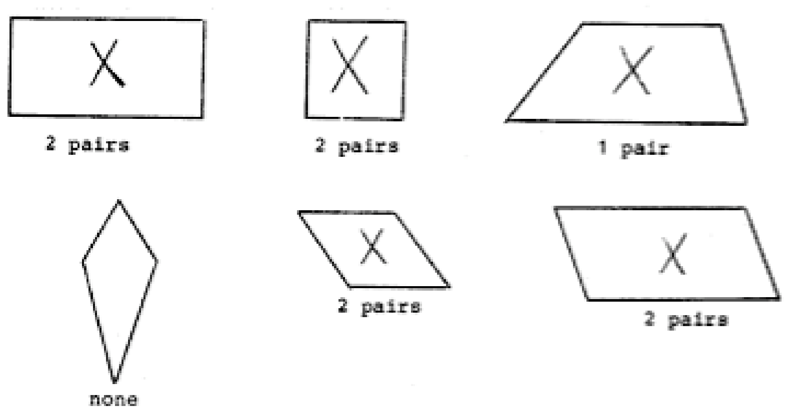
> "A quadrilateral with both pairs of opposite sides equal and all the angles equal to 90°."

After this he immediately classified a square as a rectangle, as well as seeing a rectangle as a parallelogram (one of those in the set of quadrilaterals previously called parallelograms).

Summary of results

Table 3 gives a summary of the results of all the interviews as well as how the students fared on the questions testing the Van Hiele levels in Section A. The symbols, p, pH and H used in the table, respectively represent partition, partially hierarchical and fully hierarchical thinking. With a criterion of 70% for determining proficiency for the Van Hiele levels, all fourteen cases confirm the hierarchical nature of the Van Hiele levels.

[Place table 3 more or less here]



Condition: A quadrilateral with at least one pair of opposite sides parallel (Trapeziums)

FIGURE 5. IDENTIFYING QUADRILATERALS BY MEANS OF CERTAIN DEFINING PROPERTIES

Pupil	Level 1 Recognition	Level 2 Properties	Level 3 Identification	Van Hiele level	Classification Start	Classification End
Grade 8						
Lenie	100%	90%	64%	2	P	P
Annie	43%	50%	45%	0	P	P
Rudolf	100%	90%	64%	2	P	P
Jaap	71%	40%	36%	1	P	P
San	14%	60%	36%	0	P	pH
Jan	71%	80%	45%	2	pH	H
Carin	100%	40%	45%	1	pH	H
Grade 9						
Marie	71%	80%	45%	2	H	H
Gert	29%	20%	27%	0	P	P
Dolf	71%	40%	55%	1	P	H
Gawie	43%	-	-	0	P	pH
Grade 10						
Coert	100%	90%	45%	2	H	H
Dina	100%	60%	36%	1	H	H
Nellie	86%	80%	73%	3	H	H

TABLE 3. SUMMARY OF RESULTS OF MALAN's STUDY

Findings and Conclusions

The following are some of the findings and conclusions of Malan's study:

i) Language plays an extremely important part in a child's understanding, ability and acceptance of class inclusions. It appeared from the interviews that childrens' difficulty with the hierarchical classification of quadrilaterals often lay with the meaning of the word "is" in the question "Is a square a rectangle?" They seemed to interpret it as meaning "equivalent to" or "is the same as", which of course is not what we mean by it, namely, "is a subset of" (San: "then there would be no difference between them"). This different view on the meaning of the word "is" as we use it, was possibly also manifested in their objection to using "two different names" for the same object. (Dolf: "... why do we have two names if we can say a rectangle is a parallelogram?") It seemed that the use of the word "special" for indicating class inclusions in many cases, helped students to see that we were not asking if they were equivalent, but if the one was a subset of the other (see Dolf).

ii) The use of analogy in referring to other situations where an object may have two different names, or in other words, where it may be viewed as a special subset of a larger set, sometimes seemed useful. Also when a student had already made a hierarchical inclusion of some quadrilaterals, it provided the opportunity to convince him/her of the functionality of repeating it with other sets of quadrilaterals (see Carin).

iii) Tables of properties comparing quadrilaterals may be useful, but at times may reinforce partition thinking (Lenie). It seemed as if working from their own definitions for a specific quadrilateral, sometimes provided a useful starting point for convincing them of class inclusions (San, Carin). To achieve this, it was necessary to start with the more general inclusive concept, using a table only when their definition was incomplete (e.g. insufficient).

iv) Deductive thinking was already in evidence among children at levels lower than Level 3. For instance, Lenie provides a good example when she uses deduction to justify her exclusion of rectangles from the parallelograms. This is in support of Njisane's earlier reported findings, and partially justifies our theoretical misgivings with an oversimplified view of deduction and its role in mathematics.

v) The theoretical claim that (informal) deductive thinking develops hand in hand with hierarchical class inclusion seems doubtful. Whereas Lenie uses deduction to justify her partition thinking, others like Carin and San could reason logically from their own or given definitions to make class inclusions. Although conditional (deductive) reasoning with class inclusions (all p are q) and causal relationships (p implies q) are logically equivalent (e.g. $(A \subset B;\ B \subset C;\ \ A \subset C) <=> (p => q;\ q => r;\ \ p => r))$, it does not necessarily mean that they are experienced as psychologically the same by children. A study by O'Brien (1973) has, for instance, demonstrated that causal arguments (in non-mathematical situations) are consistently easier than class inclusions.

vi) Bearing in mind that hierarchical thinking is only supposed to emerge on Level 3 according to (some interpretations of) the Van Hiele theory, it is surprising, and in fact contradictory, that so many students at lower levels could be led to hierarchical classification. The same can be said of Van Hiele's assertion in **Begrip en Inzicht** that it first occurs on Level 2. Lenie and Rudolf at Level 2, however, complicate matters by refusing to make any inclusions.

vii) Furthermore, from the data it seems that hierarchical thinking is far more dependent on the teaching strategy used, than on the Van Hiele level as measured by the test-items in Section A (recall that the test items did not directly test hierarchical classification). This independence was particularly highlighted by the success of the teaching strategy used for Gawie, where the learning events were structured in such a manner that hierarchical relationships between the figures were immediately put in the foreground.

viii) We believe that there is great danger in the Van Hiele based argument that pupils should first learn the properties of quadrilaterals before any attention is given to the making of hierarchical inclusions. If the learning of the properties of each quadrilateral is done in isolation of each other, and they are not continually compared, children's tendency to partition may fossilise and they may use precisely those learnt properties to justify their partitioning (e.g. Lenie and Rudolf). A partition classification schema once firmly entrenched therefore seems very resistant to change. In contrast, those who did not possess such firmly entrenched prior ideas (generally those at lower Van Hiele levels) could more easily be led to class inclusions via appropriate teaching strategies (San, Carin, Gawie, Dolf).

ix) Given the contradictory nature of the evidence of Malan's study, as well as in relation to Njisane's study, we have hypothesized that the hierarchical class inclusion of mathematical objects may develop independently from the development of deductive thinking. In our view they neither necessarily develop simultaneously, nor does the one need to be considered a prerequisite for the other.

An alternative teaching strategy

Description

Another aspect of Malan's study was the testing of an alternative teaching strategy (similar to the one used for Gawie) with a group of five Grade 6 students. They were chosen since they had not yet been "contaminated" by the traditional approach. Although they knew the names for a square and a rectangle, they did not know any of their properties nor had they seen or heard the names of the other quadrilaterals.

The students were first told that a quadrilateral was any closed figure with four sides and that one could obtain/construct special quadrilaterals if one placed restrictions on certain properties. Then unfamiliar properties which quadrilaterals could possess like parallel sides, perpendicular diagonals, opposite sides, opposite angles, etc. were clarified without referring to specific quadrilaterals. They were then invited to select any such properties they could think of. The first one they chose was that both pairs of opposite sides must be equal. They were then told to construct (draw) all the quadrilaterals which complied with that description. Initially they first drew just a square and a rectangle, but the researcher then drew a (skew) rhombus on the blackboard, asking them if that complied with the description. They then soon discovered a (skew) parallelogram by themselves.

The same procedure was then repeated for other properties (e.g. diagonals are equal, all the sides are equal, opposite angles equal, and sides parallel, etc.). The various descriptions and their corresponding visual prototypes were then summarised on the blackboard and the students were then asked questions like the following:

a) Are all the figures with property A present in the figures with property B?

b) Are all the figures with property B present in the figures with property A?

c) Are there cases where different properties have the same set of figures?

From all five students' positive responses to these questions (which were afterwards verified individually), it was clear that this strategy promoted both hierarchical classification, as well as tolerance for alternative, but logically equivalent definitions (e.g. question (c) above).

This alternative approach is clearly radically opposed to the traditional approach where children first learn to associate the names of figures with given visual prototypes. The defining quality associated with the name is therefore determined by the visual perception of the figure. Since a square "looks" different from a rectangle, the traditional approach forces children into partition classification right from the start. We believe that the observation that children think of shapes as a whole without explicit reference to their components, is the direct result of our actually teaching children from the start to think of shapes as a whole and in terms of visual prototypes, and with no reference to their components. However, in the alternative approach whole sets of figures are associated with specific defining properties. For example, since squares, rhombi and kites are grouped together under one or more defining properties, hierarchical classification is clearly promoted. A comparison between the traditional and this alternative approach is given in Table 4. Summarised: in the traditional approach, properties are introduced via certain geometric figures, while in this alternative approach geometric figures are introduced via their defining properties. Interestingly, this approach may not be as unnatural to children as it looks at first sight, since Senk (1983: 163) found that the largest number of misfits of the Van Hiele model, were those that had mastered Level 2 without mastering Level 1.

[Place table 4 more or less here]

We have also hypothesized that students in such an alternative approach will probably progress through levels quite different from the normal Van Hiele levels: the latter having being theoretically developed from the traditional approach. What the precise characteristics of these levels will be, is a matter for urgent future research. In the final analysis, it therefore seems from our investigation that the Van Hiele theory and its levels are not immutable and totally independent of the teaching strategies used. Mayberry (1981: 8) seems to have been justified when suggesting: "It is conceivable that the observed levels are an artifact of the curriculum or of the instruction given the students ..."

Perhaps the Van Hiele levels should not so much be viewed as prescriptive in regard to a learning and teaching hierarchy, but merely as descriptive of the results and outcomes of our present teaching strategies and curricula. It is therefore possibly only "prescriptive" in so far as the traditional approach is used.

Some mathematical and educational perspectives

Some mathematics educators may, however, object to the alternative approach, arguing that it pre-empts the definition of geometric figures, thereby robbing students of the opportunity to construct them for themselves. It also circumvents the transition from partition to hierarchical classification. Certainly from a mathematical point of view,it is true that mathematicians often need to define objects "a posteriori", as well as to make a hierarchical transition in their conceptual view of objects on the grounds of economy (especially economy of definition). These activities of defining and classifying are furthermore pedagogically important, because they lead to the construction of powerful alternative conceptual schemas, a process comparable to any major scientific breakthrough. To deny students these opportunities, is to deny them important educational experiences.

On the other hand, there are several other areas like the set of real numbers and its subsets which could provide an easier transition from partition to hierarchical classification. Furthermore, mathematicians sometimes define mathematical objects "a priori" by the variation of the properties

TRADITIONAL APPROACH	ALTERNATIVE APPROACH
1. Children become visually familiar with the various geometric figures and their names, e.g.: quadrilaterals, like squares, rectangles, parallelograms, etc.	1. Children are visually familia-rised with the properties of geometric figures, e.g.: para sides, equal angles, bisectin diagonals, etc.
2. Children now analyse these visually familiar figures separately to discover their properties.	2. Now variations of these proper are analised. Children select certain properties to act as constraints, and then try and construct all those geometric figures which comply with thos conditions.

TABLE 4. A COMPARISON OF TEACHING APPROACHES

of known objects, and are then faced with the task of constructing examples complying with those conditions. For example, defining a "skew kite" by the variation of the normal definition of a kite to "a quadrilateral with at least one pair of adjacent sides equal" (De Villiers, 1986a: 33). In addition, the alternative approach has the positive aspect that children will accept class inclusions painlessly, thereby allowing more time for developing the notion of proof and deduction.

Another valid point of objection to the alternative approach is that children will not be coming into the classroom completely untainted, since the social environment would already in many pre-school children have them associate, for instance, a rectangle with a specific visual prototype (not all sides equal). They would therefore already be conditioned into the partitioning of figures beforehand, and would resist such an alternative approach. However, this would hardly be the case for most other quadrilaterals like rhombi, parallelograms and trapezia. And since using the alternative approach would enable children to at least easily make some class inclusions, one may use analogy from there to convince and enable them to see a square as a rectangle (see Carin).

SOME DRAWBACKS OF THESE STUDIES

First of all, in Malan's study it is possible, but probably unavoidable, that the Hawthorne effect played a significant role. Also it was impossible to assess any changes that had occurred to their Van Hiele level thinking as measured by the questionnaire because of and during the interview-teaching experiment. Furthermore, since there was no follow-up interview at a later stage, we do not know whether the transition to hierarchical classification had been permanent. The results are also difficult to generalize to normal classrooms, since the situation was either an ideal one-to-one teaching environment or a small group.

An obvious point of valid criticism against both studies could be that the results are unique to our South African situation, our teaching methods and curriculum, and not suitable for generalization to other countries. However, a close analysis in Usiskin (1982) of American students' fall performances on items 13 and 14 (testing hierarchical inclusion), at 26% and 13% respectively, compare unfavourably with the other three items (measuring informal deduction) at 48%, 43% and 30%. This clearly supports Njisane's finding that hierarchical classification (of quadrilaterals) seems psychologically more difficult to children than deduction in general. In another local study presently underway, Smith (1987) has found similar results using a slightly adapted (and translated) version of the CDASSG-test in Usiskin (1982).

That children at lower Van Hiele levels may achieve success in proofwriting, has also been confirmed by Senk (1983: 179) when she found that 23% of the Level 2 students could write at least 3 of 4 valid proofs. Shaughnessy & Burger (1985: 423) have also reported that "class inclusions were seldom recognized without a lot of probing ..." and that "... most students balked at the suggestion that these figures had more than one name ..."

SOME GENERAL COMMENTS ABOUT TESTING HIERARCHICAL CLASSIFICATION

Although children at an early age are capable of understanding class inclusions like "cats and dogs are animals", it is certainly psychologically much more difficult with geometric figures, since the defining attributes are usually more subtle and complex. Class inclusions among different classes of geometric figures are also not necessarily of the same psychological difficulty, although the logical structure may be the same. For instance, in Mayberry (1981) only 3 out of 19 students indicated the squares also as rectangles on a sheet of some given quadrilaterals, while 12 out of 19 indicated an equilateral triangle also as an isosceles triangle in a comparable task. Psychologically, it is easy to explain this discrepancy since the visual prototypes used for equilateral triangles are easily visually recognizable as isosceles by, for instance, mentally folding the equilateral triangle along any line of symmetry. In contrast, it is not quite so easy visually recognizing a square as a rectangle. In addition, it must be remembered that equilateral and isosceles triangles are usually introduced with verbal definitions, while that is usually not the case with a square and a rectangle. (Actually Mayberry (1981) has already confirmed that students do not necessarily simultaneously progress to the same level of Van Hiele thinking (in general) by doing a consensus analysis of different conceptual strands.)

An analysis of student responses to questions measuring hierarchical classification in Njisane (1986) by means of visual identification (given a sheet of quadrilaterals) or by means of a verbal description (all rectangles are parallelograms), indicates that students consistently perform lower on visual identification tasks than on verbal ones. This can once again be explained by the misleading nature of visual identification tasks, where the student may only mark the most general example of that quadrilateral, not knowing that the intention of the question was that he should also mark the special cases. Even mathematics teachers who know the quadrilateral inclusions, are easily trapped by such a task, unless they are for instance, asked if any of the other figures can also be marked as special cases of that quadrilateral. The possibility also exists that the level of difficulty of class inclusion varies quite significantly even among the quadrilaterals. For example, some students (see San) sometimes easily recognize a rhombus as a kite by (presumably) mentally rotating it into the standard representation of a kite (perhaps they recognize the perpendicularity of the diagonals and (mistakenly) use it as a sufficient condition). In contrast, it seems that the inclusion of squares among the rectangles, is frequently the most difficult (see Carin).

THE ROLE OF HIERARCHICAL CLASSIFICATION IN MATHEMATICS

The main reasons why mathematicians usually prefer hierarchical classifications to partitioning, are:

i) it leads to an economy of definition (e.g. compare "a parallelogram is a quadrilateral with two pairs of opposite sides equal" to "a parallelogram is a quadrilateral with unequal diagonals, two pairs of opposite sides equal, but not all sides equal")

ii) it simplifies the deductive structure of a set of concepts (e.g. defining a rectangle as a special kind of parallelogram implies that all the parallelogram theorems are immediately applicable to rectangles, without having to prove them anew)

iii) it is sometimes a useful conceptual schema when proving certain riders (e.g. proving that a kite with one pair of opposite sides parallel, is a rhombus by using the fact that a rhombus is both a kite and a parallelogram, and then merely proving that both pairs of opposite sides are parallel).

TEACHING THE HIERARCHICAL INCLUSIONS OF QUADRILATERALS IN THE TRADITIONAL APPROACH

We believe that children's difficulty with hierarchical class inclusion may not lie so much with the inclusion as such, but with the meaning of the activity: both linguistic and functional: linguistic in the sense of correctly interpreting the language used for class inclusions, and functional in the sense of understanding why it is useful. We have observed several children who have no difficulty in accepting that defining a trapezium as "a quadrilateral with at least one pair of opposite sides equal", implies that a parallelogram is also a trapezium, but that they cannot see the need for such a definition.

For hierarchical classification in the traditional approach to be really meaningful to students, it is therefore essential that not only a negotiation of linguistic meaning should take place, but also one of functional meaning: that is a discussion and exemplification of the reasons for hierarchical classification as described in the previous section. It is however not an all or nothing situation, but lies on a continuum from functional clarification on the one hand to no clarification on the other hand. However, while functional clarification and its negotiation requires as negative trade-off a lot of time and patience, the imposition of hierarchical definitions with no clarification, leads to a certain amount of didactical economy as shown in Figure 6. Therefore, even though "complete" functional understanding is conceptually desirable, a midway strategy might be preferable. Nonetheless, whether the teacher chooses to spend a lot of time on functional clarification of content and processes he (or she) should at least have a clear idea himself (herself) of the various functions of mathematical content and processes (e.g. What is it good for? Why is it necessary? What is its role? What are the reasons behind it? Does it have any applications?, etc.)

[Place figure 6 more or less here]

On the other hand, hierarchical classification of quadrilaterals is not all that important: one can get along quite well without it (except that one's definitions are then uneconomical). This raises the question: why not allow children to proceed with their partition definitions (provided they're consistent in their partitioning)? Surely it is more desirable than imposing hierarchical definitions on them which they do not understand. In fact, in the latter part of 1986, the author participated in a successful teaching experiment with a Grade 9 class where the possibility of hierarchical inclusion and hierarchical definitions as an alternative way of looking at the quadrilaterals was discussed, but the children were free to choose whichever view they preferred. The arbitrariness in choice of definition was thus emphasized in a broader context than usual. Although quite a number of children gradually saw the convenience of hierarchical inclusion and made the transition, a significant number of children (some of the most intelligent) preferred not to do so. Of course, this meant that none could be penalised in tests and exams if they used partitioning instead of class inclusion.

FINAL REMARKS

As pointed out in this paper, two aspects of the Van Hiele theory need clarification and further research, namely a refinement with regard to the levels at which deduction (as justification, explanation and systematization) is supposed to occur, as well as the relationship between hierarchical thinking and deduction. Furthermore, it is necessary to characterize children's development in the alternative approach as described earlier.

FIGURE 6. FUNCTIONAL CLARIFICATION VERSUS NO CLARIFICATION

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