Converses of Napoleon's Theorem

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Interesting converse results in elementary geometry can often be found by taking certain parts of a figure as given "in position" and investigating the extent to which various other parts of the figure are determined. In this article we use this tactic to obtain some apparently new converses of the well-known theorem of Napoleon. Geometry is more a point of view than a methodology, and we employ a variety of different arguments (synthetic, coordinate, transformational, complex analytic) to establish our results. To set the stage, we begin with an overview of Napoleon's theorem and a glimpse of its long history.

1. NAPOLEON'S THEOREM AND TORRICELLI'S CONFIGURATION. The familiar but curious theorem attributed to Napoleon Bonaparte asserts that the centers L, M, N of the three equilateral triangles $\triangle BXC$, $\triangle CYA$, $\triangle AZB$ built outwards on the sides $\overline{BC}, \overline{CA}, \overline{AB}$ of an arbitrary triangle $\triangle ABC$ are the vertices of an equilateral triangle, and the same is true of the centers L', M', N' of the three inward equilateral triangles $\triangle CX'B$, $\triangle AY'C$, $\triangle BZ'A$.

The configuration formed by a triangle, the equilateral triangles on its sides, the "Napoleon" triangles, and various connecting lines and circles (commonly called "Torricelli's configuration" a century ago), has many elegant and unexpected properties.

The outward case. Suppose (Figure 1) that $\triangle ABC$ is a positively oriented triangle (so that $A \to B \to C \to A$ is counterclockwise). The outer Napoleon triangle $\triangle LMN$ is also positively oriented, and its center coincides with the centroid G of $\triangle ABC$. Lines $\overleftrightarrow{AX}, \overleftrightarrow{BY}, \overleftrightarrow{CZ}$ are concurrent at a point F, called the *outward Fermat point* of $\triangle ABC$, and F lies on the circumcircle of each outward equilateral triangle $\triangle BXC$, $\triangle CYA$, $\triangle AZB$ and also on the circumcircle of the inner Napoleon triangle $\triangle L'M'N'$. Lines $\overleftrightarrow{AX}, \overleftrightarrow{BY}, \overleftrightarrow{CZ}$ make acute angles of 60° with each other at F, and $AX = BY = CZ = \pm AF \pm BF \pm CF$, a minus sign being taken if the angle of $\triangle ABC$ at that vertex exceeds 120°. The vertices A, B, C are symmetric to F in the sidelines $\overleftrightarrow{MN}, \overleftrightarrow{NL}, \overleftrightarrow{LM}$ of the outer Napoleon triangle $\triangle LMN$. Lines $\overleftrightarrow{AL}, \overleftrightarrow{BM}, \overleftrightarrow{CN}$ are concurrent. When $\triangle ABC$ has a 120° angle, F is the vertex of that angle; when $\triangle ABC$ has an angle larger than 120°, F lies in the angle vertical to that angle; and when every angle of $\triangle ABC$ is smaller than 120°, F lies inside $\triangle ABC$ and is the point P that solves the problem Fermat posed to Torricelli: minimize $f(P) = PA + PB + PC$. When the largest angle of $\triangle ABC$ exceeds 120°, the solution of Fermat's problem is the vertex of that largest angle.

The inward case. Analogous properties hold for the inward case. Suppose (Figure 2) that $\triangle ABC$ is a positively oriented scalene triangle. The inner Napoleon triangle $\triangle L'M'N'$ is negatively oriented, and its centroid coincides with the centroid G of $\triangle ABC$. Lines $\overleftrightarrow{AX'}, \overleftrightarrow{BY'}, \overleftrightarrow{CZ'}$ are concurrent at a point F', called

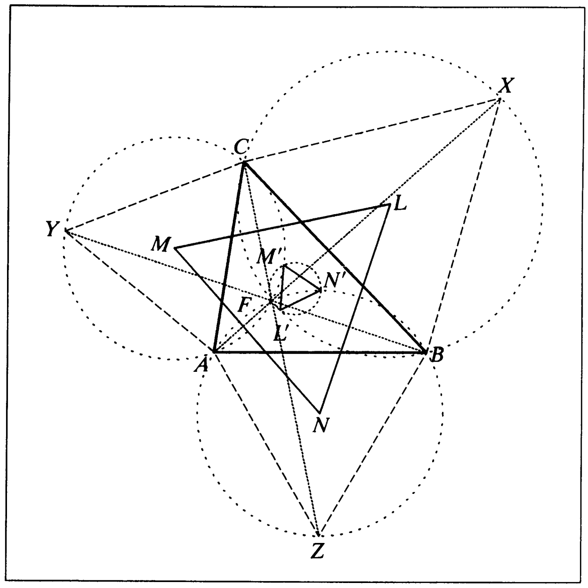

FIG. 1. The outward case.

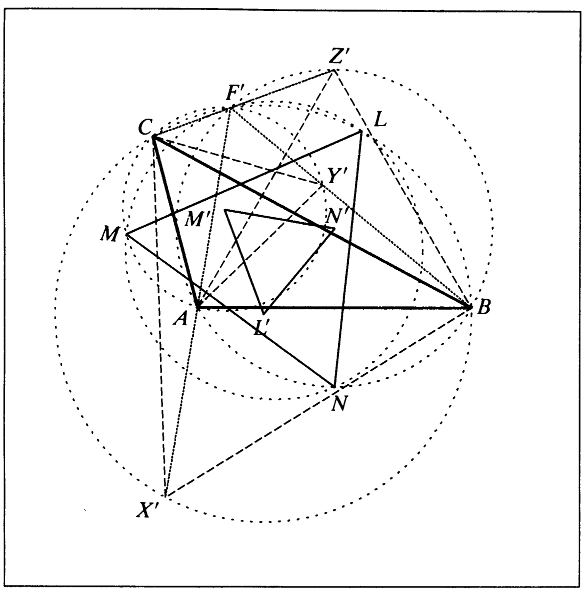

FIG. 2. The inward case.

the *inward Fermat point* of $\triangle ABC$, and F' lies on the circumcircle of each inward equilateral triangle $\triangle CX'B$, $\triangle AY'C$, $\triangle BZ'A$ and on the circumcircle of the outer Napoleon triangle $\triangle LMN$. Lines $\overleftrightarrow{AX'}, \overleftrightarrow{BY'}, \overleftrightarrow{CZ'}$ make acute angles of 60° with each other at F', and $AX' = BY' = CZ' = \pm AF' \pm BF' \pm CF'$, a minus sign being taken at each vertex where the angle of $\triangle ABC$ is larger than 60°. The vertices A, B, C are symmetric to F' in the sidelines $\overleftrightarrow{M'N'}, \overleftrightarrow{N'L'}, \overleftrightarrow{L'M'}$ of the inner Napoleon triangle $\triangle L'M'N'$, and lines $\overleftrightarrow{AL'}, \overleftrightarrow{BM'}, \overleftrightarrow{CN'}$ are concurrent. The point F' is never inside $\triangle ABC$. When $\triangle ABC$ has exactly one 60° angle, F' is that vertex; and when two angles of $\triangle ABC$ are both larger or both smaller than 60°, F' lies outside $\triangle ABC$ inside the angle at the third vertex. Refining a claim of Courant and Robbins [**4**; pp. 354–359], Brownawell and Goodman [**2**] have shown that if $\angle A \geqslant 60°$ and $\angle B \geqslant 60°$, for example, then F' is the point P that maximizes $g(P) = PC - PA - PB$. When $\triangle ABC$ has two angles less than 60°, the solution of this maximum problem is the vertex of the smallest angle.

The collinear case. Most of these properties, suitably phrased, are correct when A, B, C are collinear (FIGURE 3) and the distinction between "inner" and "outer" is lost. In this case the inner and outer pictures are symmetric in the line of collinearity.

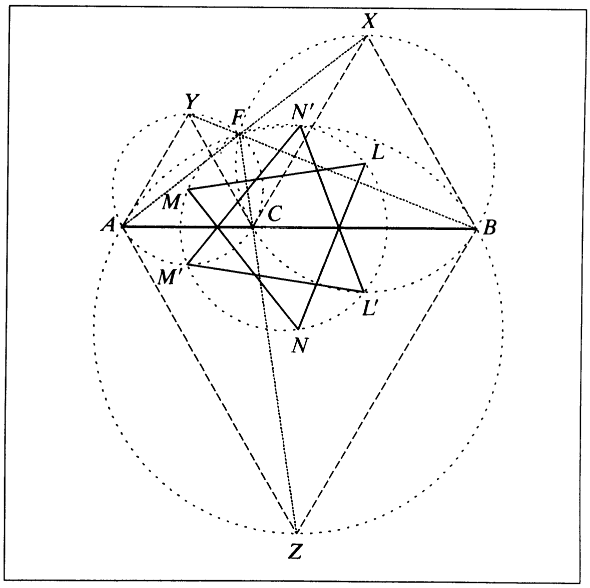

FIG. 3. The collinear case.

Some further properties. Here are a few of the many additional properties of the Torricelli configuration that appear in the early literature. Triangles $\triangle AYZ'$, $\triangle AY'Z$, etc., are congruent to $\triangle ABC$, and their circumcenters lie on the circumcircle of $\triangle ABC$. The sum of the areas of the Napoleon triangles $\triangle LMN$ and $\triangle L'M'N'$ is the average of the areas of the three outward equilateral triangles on the sides of $\triangle ABC$, and the difference of these areas is the area of $\triangle ABC$. The line $\overleftrightarrow{FF'}$ through the two Fermat points bisects the segment $\overline{HG}$ that joins the orthocenter H and the centroid G of $\triangle ABC$. The point Q so that the figure $F'HFQ$ is a parallelogram lies on the circumcircle of $\triangle ABC$. And the triangle formed by the lines through A, B, C perpendicular to $\overleftrightarrow{AF}, \overleftrightarrow{BF}, \overleftrightarrow{CF}$ is the largest equilateral triangle that can be circumscribed about $\triangle ABC$, and its area is $4(ABC)$. (These results and more can be found in Mackay [**21**].)

Finally we mention one particularly elegant recent observation (Garfunkel and Stahl [**15**]). Let A_1, A_2 be the trisection points of the side $\overline{BC}$ of $\triangle ABC$ with A_1 nearer B, and define B_1, B_2 and C_1, C_2 similarly on $\overline{CA}$ and $\overline{AB}$. Then the summits of the six outward and six inward equilateral triangles on the sides of the irregular hexagon $A_1A_2B_1B_2C_1C_2$ form concentric regular hexagons.

Sources. Napoleon's theorem is surely one of the most-often rediscovered results in mathematics. The literature is extensive and offers almost a plethora of related results, extensions, and generalizations, supported by divers arguments. Many writers have used it as a kind of touchstone to establish the efficacy of their favorite approaches to geometry. An assortment of proofs can be found in the following readily available sources: Court [**5**, pp. 105–107], Coxeter and Greitzer [**6**, pp. 60–65, 82–83], Demir [**7**], Fettis [**10**], Finney [**11**], Forder [**14**, p. 40], Garfunkel and Stahl [**15**], Honsberger [**17**, pp. 24–36, 40, 147–152], Johnson [**18**, pp. 218–224], Mauldon [**22**], Rabinowitz [**25**], Yaglom [**32**, pp. 38–40, 93–97]. Most of these references discuss related results and some properties of the full configuration. Generalizations of various kinds can be found in many of these references, and especially in, for example, Berkhan and Meyer [**1**, pp. 1216–1219], Douglas [**8**], Finsler and Hadwiger [**12**], Fisher, Ruoff, and Shilleto [**13**], Gerber [**16**], Neumann [**23**], [**24**], Rigby [**26**], and Schütte [**28**], most of which list numerous additional sources.

Why Napoleon? The early history of Napoleon's theorem and the Fermat points F, F' (which are also called the *isogonic centers* of $\triangle ABC$) is summarized in Mackey [**21**], who traces the fact that $\triangle LMN$ and $\triangle L'M'N'$ are equilateral to 1825 to one Dr. W. Rutherford [**27**] and remarks that the result is probably older. The attribution of the result to Napoleon (1769–1821) has itself been the object of study (Cavallaro [**3**], Scriba [**29**]). Mackay does not mention Napoleon, nor does any other nineteenth century reference with which I am familiar. The earliest attribution I have seen appeared in 1911 in Faifofer [**9**, p. 186], where the result, posed as Problem 494, is accompanied by the parenthetical comment, "Teorema proposto per la dimostrazione da Napoleone a Lagrange." It would be of historical interest to trace the result back to Napoleon, although as Coxeter and Greitzer [**6**, p. 63] remark, "the possibility of his knowing enough geometry for this feat is as questionable as the possibility of his knowing enough English to compose the famous palindrome, ABLE WAS I ERE I SAW ELBA."

2. CONVERSES OF NAPOLEON'S THEOREM.

Interesting converse problems arise from taking parts of the Torricelli configuration as given and trying to determine the range of variability of the remaining parts of the figure. For example, one can consider existence and uniqueness questions concerning the "progenitor" triangle $\triangle ABC$ when some of the derived points $X, Y, Z, X', Y', Z', L, M, N, L', M', N', F, F'$ are prescribed. There are many possibilities, ranging from trivial to quite involved. In the following sections we consider several such converse questions.

The earliest result of this kind of which I am aware is a construction problem posed in 1868 by E. Lemoine [**20**]: Construct the triangle, given the summits of the equilateral triangles built on its sides.

An elegant construction for Lemoine's problem was provided the following year by L. Kiepert [**19**]. Points X, Y, Z are given (FIGURE 4), to be the summits of equilateral triangles $\triangle XBC$, $\triangle AYC$, $\triangle ABZ$. Let P, Q, R be the summits of the outward equilateral triangles on the sides of $\triangle XYZ$. Then A, B, C are the midpoints of $\overline{XP}, \overline{YQ}, \overline{ZR}$, respectively. Kiepert's argument (repeated in Wetzel [**31**]) uses Ptolemy's theorem and properties of the Fermat point $F = \overleftrightarrow{XP} \cap \overleftrightarrow{YQ} \cap \overleftrightarrow{ZR}$. A perspicuous motion proof can also be given. Write W_θ for the rotation about a point W through the (trigonometric) angle θ, write arguments on the left, and compose motions from the left. Then (FIGURE 4) the motion $Y_{60}X_{60}Z_{60}$ fixes A and consequently is halfturn about A. But $PY_{60}X_{60}Z_{60} = ZX_{60}Z_{60} = QZ_{60} = X$. Thus A is the midpoint of $\overline{PX}$.

In 1956, in an article whose principal objective was to promote the use of motions in the teaching of geometry, H. G. Steiner [**30**] used motions to show that

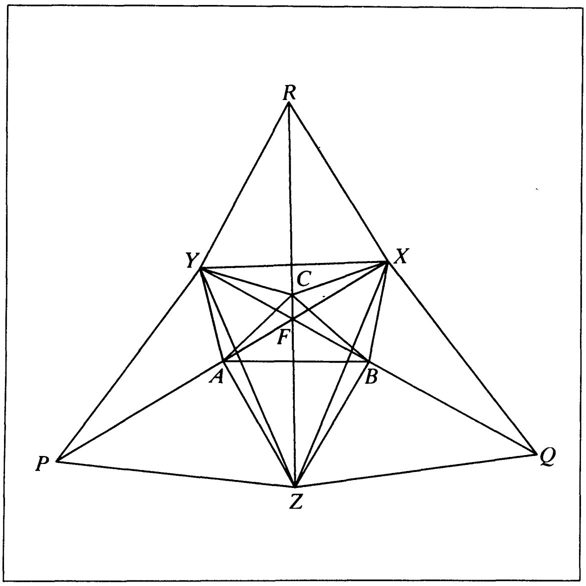

FIG. 4. Kiepert's construction.

if three distinct points X, Y, Z are given, there are, in general, *eight* triangles $\triangle ABC$ so that the triangles $\triangle ABX$, $\triangle BCY$, $\triangle CAZ$ are equilateral.

Steiner's elegant motion argument is as follows. If each of α, β, γ is $\pm 60°$, then each of the eight motions $Z_\gamma Y_\beta X_\alpha$ is a halfturn or a rotation through $\pm 60°$, and in every case it has a unique fixed point A. Defining $C = AZ_\gamma$ and $B = CY_\beta$, we see that $A = AZ_\gamma Y_\beta X_\alpha = CY_\beta X_\alpha = BX_\alpha$; and consequently triangles $\triangle CAZ$, $\triangle BCY$, $\triangle ABX$ are all equilateral. On the other hand, if $\triangle ABC$ is such a triangle, it is clear from a sketch that A is the fixed point of one of these eight motions, the signs depending on the relative orientations of $\triangle ABC$, $\triangle ABX$, $\triangle BCY$, $\triangle CAZ$. (The complicated question of whether $\triangle ABX$, $\triangle BCY$, $\triangle CAZ$ turn out to be outward or inward on the sides of $\triangle ABC$ is considered in Wetzel **[31]**.)

3. A CONVERSE OF NAPOLEON'S THEOREM. Suppose the two Napoleon triangles $\triangle LMN$ and $\triangle L'M'N'$ are given "in position." Is $\triangle ABC$ determined? We show that it is, provided that $\triangle LMN$ and $\triangle L'M'N'$ have the same center.

Obviously there is *at most one* generating triangle $\triangle ABC$, because the sidelines a, b, c of $\triangle ABC$ must be the mediators (i.e., the perpendicular bisectors) of the segments $\overline{LL'}, \overline{MM'}, \overline{NN'}$. The existence is a little more trouble. The core of the argument is the following lemma, for which we give first a traditional synthetic proof and then an argument that uses coordinates.

Lemma 1. *Five points X, X', Y, Y', T are arranged so that $\angle X'TY' = 120°$, $\angle YTX = 120°$, $XT = YT$, $X'T = Y'T$, and $XT \neq X'T$* (FIGURE 5). *Then the mediators of the segments $\overline{XX'}$ and $\overline{YY'}$ meet at a point S, and $\triangle SXX'$ and $\triangle SYY'$ are both equilateral.*

Proof: A rotation of 120° about T carries X' to Y' and Y to X, so $\overleftrightarrow{X'Y}$ and $\overleftrightarrow{XY'}$ meet at 60° at a point W. The circles through X, X', W and Y, Y', W meet at W and again at a second point S. Then $\angle XSX' = \angle XWX' = 60° = \angle Y'WY =$

$\angle Y'SY$, and so $\angle XSY' = \angle X'SY$. Since $\angle SY'X = \angle SYX'$ and $XY' = YX'$, $\triangle XSY' \cong \triangle X'SY$. Hence $SX = SX'$ and $SY = SY'$. ■

This synthetic proof, in the classical tradition, assumes that the points are positioned as in the figure. Similar arguments can, of course, be given in the various other cases, but we prefer instead to rely on a short computational proof using coordinates that is unexceptionable.

Second Proof. Introduce coordinates so that T is the origin and X' and Y' have coordinates $(2,0)$ and $(-1, \sqrt{3})$. If X has coordinates $(2s, 2t)$ with $s^2 + t^2 \neq 1$, then Y has coordinates $(-s + \sqrt{3}t, -\sqrt{3}s - t)$. Consequently the mediators of $\overline{XX'}$ and $\overline{YY'}$ have equations

$$(1-s)x - ty = 1 - s^2 - t^2$$

$$(s - \sqrt{3}t - 1)x + (\sqrt{3}s + t + \sqrt{3})y = 2(1 - s^2 - t^2);$$

and these two lines meet at a point S with coordinates $(s + \sqrt{3}t + 1, -\sqrt{3}s + t + \sqrt{3})$. A calculation confirms that $SX = XX' = X'S$ and $SY = YY' = Y'S$ irrespective of the values of s and t. ■

A point S so that both $\triangle SXX'$ and $\triangle SYY'$ are equilateral exists even when $XT = X'T$, but then the mediators of $\overline{XX'}$ and $\overline{YY'}$ coincide. Note that according to Napoleon's theorem the circumcenters of $\triangle SXX'$, $\triangle SYY'$, $\triangle XWY$, $\triangle X'WY'$ (marked in FIGURE 5) form a $60°$ rhombus.

Finally, here is our first converse of Napoleon's theorem.

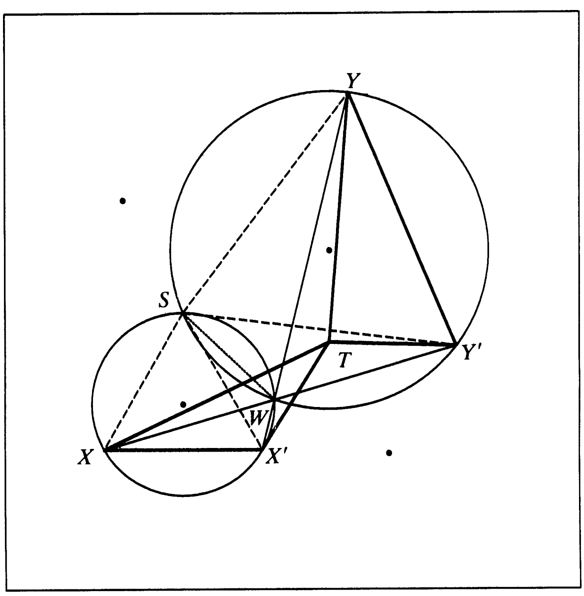

FIG. 5. An essential lemma.

Theorem 2. *Two equilateral triangles $\triangle LMN$ and $\triangle L'M'N'$ are given in position so that*

(a) *$\triangle LMN$, $\triangle L'M'N'$ are oppositely oriented and have the same center G, and*
(b) *$LM > L'M'$.*

Then there is exactly one triangle $\triangle ABC$ having $\triangle LMN$ as its outer Napoleon triangle and $\triangle L'M'N'$ as its inner Napoleon triangle, and its sides are the mediators of $\overline{LL'}, \overline{MM'}, \overline{NN'}$.

Proof: Suppose without loss of generality that $\triangle LMN$ is positively oriented. Taking the points X, X', Y, Y', T in Lemma 1 to be N, N', M, M', G, we conclude that the mediators b and c of $\overline{MM'}$ and $\overline{NN'}$ meet in a point A so that $\triangle AMM'$ and $\triangle ANN'$ are both equilateral. Similarly, if a is the mediator of $\overline{LL'}$, $B = c \cap a$, and $C = a \cap b$, then $\triangle BNN'$, $\triangle BLL'$, $\triangle CLL'$, $\triangle CMM'$ are all equilateral. The points A, B, C are different and non-collinear. (If two of the three points A, B, C were coincident, then the three mediators a, b, c would be concurrent and all three points would coincide. Then the $\pm 60°$ rotation about C that carries L to L' would carry M' to M (otherwise M would go to M' and $LM = L'M'$, contrary to (b)). This same rotation carries N to N' or N' to N, and correspondingly $LN = L'N'$ or $M'N' = MN$, contrary to (b) again. It follows that B and C, C and A, A and B lie on opposite sides of $\overleftrightarrow{LL'}, \overleftrightarrow{MM'}, \overleftrightarrow{NN'}$. Consequently, L, L', M, M', N, N' are the centers of equilateral triangles on sides $\overline{BC}, \overline{CA}, \overline{AB}$ of $\triangle ABC$. Thus $\triangle LMN$ and $\triangle L'M'N'$ are Napoleon triangles of $\triangle ABC$, and (according to (b)) $\triangle LMN$ is outer and $\triangle L'M'N'$ is inner. ■

The conclusion of the theorem is true when the inner Napoleon triangle $\triangle L'M'N'$ collapses to a point. It is also worth mentioning that any two vertices of one Napoleon triangle with any vertex of the other completely determine both Napoleon triangles "in position," so that according to the theorem they determine $\triangle ABC$ uniquely provided only that $\triangle L'M'N'$ is smaller than $\triangle LMN$.

Some consequences. The figure formed by two oppositely oriented concentric equilateral triangles has many nice properties that seem not so easy to prove without the superstructure provided by Theorem 2. Indeed, all the properties described in Section 1 have counterparts in this figure. Here are a few specific examples.

Corollary 3. *Two oppositely oriented concentric equilateral triangles $\triangle LMN$ and $\triangle L'M'N'$ are given, with $L \neq L'$, $M \neq M'$, and $N \neq N'$. Then*

(a) *Lines $\overleftrightarrow{LL'}, \overleftrightarrow{MM'}, \overleftrightarrow{NN'}$ lie in a pencil. They are parallel if $LM = L'M'$ and concurrent otherwise.*
(b) *The points $\overleftrightarrow{LM} \cap \overleftrightarrow{L'M'}, \overleftrightarrow{MN} \cap \overleftrightarrow{M'N'}, \overleftrightarrow{NL} \cap \overleftrightarrow{N'L'}$ are collinear (one might be at infinity).*
(c) *The centroid of the triangle $\triangle A_0B_0C_0$ of midpoints A_0, B_0, C_0 of $\overline{LL'}, \overline{MM'}, \overline{NN'}$ is at the common center G of the two given triangles.*
(d) *Suppose that $LM \neq L'M'$, and let A be the point in which the mediators b, c of $\overline{MM'}, \overline{NN'}$ intersect. Then the points S, S' symmetric to A in $\overleftrightarrow{MN}, \overleftrightarrow{M'N'}$ lie on the circumcircles of $\triangle L'M'N'$, $\triangle LMN$.*

Proof: (a) It is easy to verify directly that the lines are parallel if $\triangle LMN$ and $\triangle L'M'N'$ have the same circumcircle. If $LM > L'M'$, then $\overleftrightarrow{LL'}, \overleftrightarrow{MM'}, \overleftrightarrow{NN'}$ are the mediators of the sides of $\triangle ABC$ and hence are concurrent at the circumcenter of that triangle.

(b) This follows from (a) by Desargues' theorem.

(c) When $LM > L'M'$, the common center G of $\triangle LMN$ and $\triangle L'M'N'$ is the centroid of $\triangle ABC$ and consequently also the centroid of its medial triangle $A_0B_0C_0$. The result when $LM = L'M'$ follows by continuity, for example.

(d) Points S, S' are the Fermat points of $\triangle ABC$. ■

4. ANOTHER CONVERSE. To what extent is the progenitor triangle $\triangle ABC$ determined if only one Napoleon triangle is given in position? The answer to this question is a little more complicated.

Suppose $\triangle PQR$ is a given equilateral triangle, to play the role of $\triangle LMN$ *or* $\triangle L'M'N'$. Taking our cue from FIGURES 1, 2, and 3, we generate the vertices A, B, C by reflecting a point S in the lines $\overleftrightarrow{QR}, \overleftrightarrow{RP}, \overleftrightarrow{PQ}$ (FIGURE 6). Since $PB = PS = PC$, $\triangle BPC$ is isosceles, and it is easy to see that $\angle CPB = 2\angle QPR = \pm 120°$ by summing angles at P. Consequently P is the center of an equilateral triangle built on $\overline{BC}$. Similarly for Q and R, of course, and since $\triangle PQR$ is equilateral it follows that it is a Napoleon triangle of $\triangle ABC$. The problem is to determine when $\triangle PQR$ is inner and when it is outer. Here is our second converse.

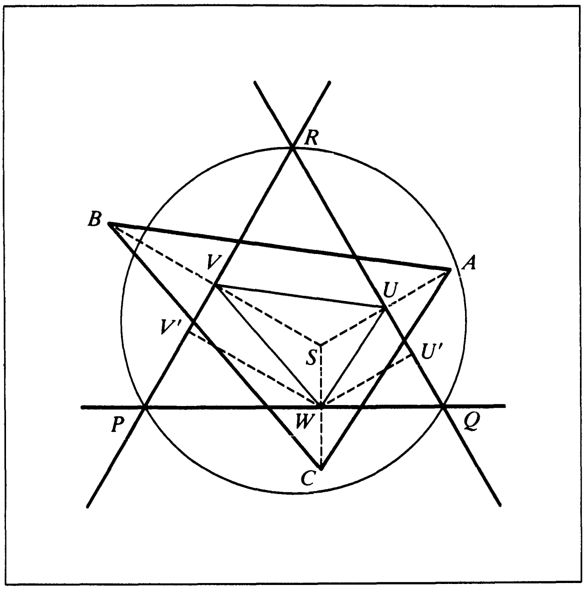

FIG. 6. The case of one given Napoleon triangle.

Theorem 4. *Let $\triangle PQR$ be a positively oriented equilateral triangle with circumcircle Γ, and for any point S let A, B, C be the points symmetric to S in $\overleftrightarrow{QR}, \overleftrightarrow{RP}, \overleftrightarrow{PQ}$. Then:*

(a) *When S lies on Γ, points A, B, C are collinear.*

(b) *When S lies inside Γ, then $\triangle PQR$ is the outer Napoleon triangle of $\triangle ABC$, and S is its outward Fermat point. The largest angle of $\triangle ABC$ is greater than, equal to, or less than 120° according to whether S lies outside, on, or inside $\triangle PQR$.*

(c) *When S lies outside Γ, then $\triangle PQR$ is the inner Napoleon triangle of $\triangle ABC$, and S is its inward Fermat point. One angle of $\triangle ABC$ is a 60° angle precisely when S lies on a sideline of $\triangle PQR$, and $\triangle ABC$ has two angles larger than 60° when S lies in one of the regions off a vertex of $\triangle PQR$ and two angles smaller than 60° when S lies in one of the regions off an edge of $\triangle PQR$.*

Proof: Let U, V, W be the feet of the perpendiculars from S to $\overleftrightarrow{QR}, \overleftrightarrow{RP}, \overleftrightarrow{PQ}$. Since a dilatation with center S and ratio 2 carries U, V, W to A, B, C, it is enough to study triangle $\triangle UVW$.

The fact that U, V, W are collinear if and only if S lies on the circumcircle Γ is well known, and the line on which they lie is the Simson line of S. (See Coxeter and Greitzer **[6]** for an exposition of this classical theory.)

When S moves, the orientation of its pedal triangle $\triangle UVW$ remains unchanged unless the points U, V, W become collinear, which occurs only when S lies on Γ. It follows that the orientation of $\triangle UVW$, and so of $\triangle ABC$, agrees with that of $\triangle PQR$ for S inside Γ and is opposite that of $\triangle PQR$ for S outside Γ. Consequently $\triangle PQR$ is the outer Napoleon triangle of $\triangle ABC$ when S lies inside Γ and the inner Napoleon triangle of $\triangle ABC$ when S lies outside Γ.

Now suppose S lies inside Γ, and suppose (with no loss of generality) that $\angle UWV$ is a maximal angle of $\triangle UVW$. (Since the sides of $\triangle UVW$ are proportional to the distances PS, QS, RS (see Coxeter and Greitzer **[6**, p. 23]), this requires only that S lie in the sector PGQ, where G is the center of $\triangle PQR$.) Let U', V' be the feet of the perpendiculars from W to $\overleftrightarrow{QR}, \overleftrightarrow{RP}$. Then $\angle U'WV' = 120°$, and it is apparent that $\angle UWV$ is less than, equal to, or greater than $120°$ according to whether S lies inside, on, or outside $\triangle PQR$.

A similar argument can be given in the inward case (c); we omit the minutiæ. ■

The assertion can be phrased more symmetrically. If $\Gamma_P, \Gamma_Q, \Gamma_R$ are the images of Γ under reflection in $\overleftrightarrow{QR}, \overleftrightarrow{RP}, \overleftrightarrow{PQ}$, then $\triangle PQR$ is the outer Napoleon triangle of $\triangle ABC$ when A, B, C lie inside $\Gamma_P, \Gamma_Q, \Gamma_R$ (respectively) and the inner Napoleon triangle of $\triangle ABC$ when A, B, C are outside $\Gamma_P, \Gamma_Q, \Gamma_R$.

I am indebted to G. D. Chakerian for much of this elegant geometric argument, contained in a letter dated November 28, 1979. My original argument employed coordinates. In another letter dated January 2, 1980, Chakerian observed that parts of the theorem follow immediately from the formula

$$(ABC) = \frac{3\sqrt{3}}{4}(r^2 - \rho^2)$$

for the signed area of $\triangle ABC$ in terms of the radius r of Γ and $\rho = GS$ (see, for example, Johnson **[18**, p. 139]).

In summary, we have the following:

Corollary 5. *A progenitor triangle exists for a given equilateral triangle* $\triangle$LMN *and Fermat point F precisely when F lies inside the circumcircle of* $\triangle$LMN, *and then it is unique. A progenitor triangle exists for a given equilateral triangle* $\triangle$L′M′N′ *and Fermat point F′ precisely when F′ is outside the circumcircle of triangle* $\triangle$L′M′N′, *and then it is unique.*

Exercise: What is the story if L, M, N, and F' (or L', M', N', and F) are given?

5. MIXED CONVERSES. Finally we consider two similar-looking converse situations for which the results turn out to be surprisingly different.

The case X, Y, N. When X, Y, N are prescribed, we shall see that again there is a significant circle. An analysis using motions gets us started. In Figure 1 it is plain that the motion $Y_{60}X_{60}N_{120}$ fixes A, and consequently it must be A_{240}. Suppose conversely that points P, Q, R are given (to play the role of X, Y, N). The motion

$Q_{60}P_{60}R_{120}$, being a 240° rotation, has a unique fixed point A. Let $C = AQ_{60}$ and $B = CP_{60}$. Then $AQ_{60}P_{60}R_{120} = CP_{60}R_{120} = BR_{120}$, so $\triangle ACQ$ and $\triangle CBP$ are equilateral with $\angle AQC = \angle BPC = 60°$ and $\triangle BAR$ is isosceles with $\angle BRA = 120°$.

Under what circumstances are A, B, C collinear? And if they are not collinear, when are P, Q, R the points X, Y, N of $\triangle ABC$ and when are they X', Y', N'? In other words, when is $\triangle ABC$ positively oriented and when negatively? Here is the result.

Theorem 6. *Distinct points P, Q, R are given (to play the role of X, Y, N). Let S be the point so that $\triangle PQS$ is a positively oriented equilateral triangle, and let T be the center of $\triangle PQS$. Let Γ be the circle through T with center S. Then there exists a unique triple ABC so that $\triangle BCP$ and $\triangle CAQ$ are equilateral and $\triangle CAR$ is isosceles with $\angle R = \pm 120°$; and (see* FIGURE 7):

(a) *If R lies on Γ, points A, B, C are collinear;*
(b) *If R lies inside Γ, $\triangle ABC$ is positively oriented, and its points X, Y, N are the given points P, Q, R;*
(c) *If R lies outside Γ, $\triangle ABC$ is negatively oriented, and its points X', Y', N' are the given points P, Q, R.*

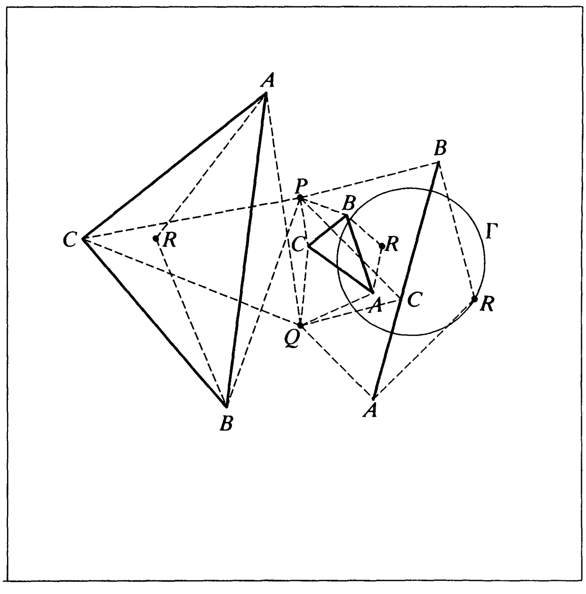

FIG. 7. The case X, Y, N.

Proof: Matters such as these are easily handled in the complex plane. Introduce coordinates so that the points P, Q, and T have complex coordinates 1, 0, and $d = 1/2 - (\sqrt{3}/6)i$, respectively, and write $h = e^{i\pi/6}$. Recall that in the complex plane, rotation through the (trigonometric) angle θ about a point z' is given by the linear mapping $w = z' + e^{i\theta}(z - z')$. If R has complex coordinate z_0, we see by composing the mappings that the key motion $Q_{60}P_{60}R_{120}$ is given by the transformation $w = h + (1 + \bar{h})z_0 - hz$. The coordinates a, c, b of the fixed point A of this transformation and of the points $C = AQ_{60}$ and $B = CP_{60}$ are easy to compute: $a = d + \bar{h}z_0$, $b = hd + z_0$, and $c = -hd + hz_0$. The signed area (ABC) of $\triangle ABC$ in the complex plane whose vertices have coordinates a, b, c is given by the determinant

$$(ABC) = \frac{i}{4}\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} = -\frac{1}{2}\Im m\left(a\bar{b} + b\bar{c} + c\bar{a}\right),$$

and for the case at hand a short calculation shows that $(ABC) = -\frac{1}{4}\sqrt{3}\left(|z_0 - \bar{h}|^2 - \frac{1}{3}\right)$. Now the claims of the theorem are easily checked. ∎

To summarize, if two different points U, V are given, let $\triangle UVS$ be a positively oriented equilateral triangle and let $\Gamma(U, V)$ be the circle with center S that passes through the center T of $\triangle UVS$. Then we have the following Napoleon converse.

Corollary 7. *A progenitor triangle exists for given X, Y, N precisely when N lies inside the circle $\Gamma(X, Y)$, and it is unique; and a progenitor triangle exists for given X', Y', N' precisely when N' lies outside the circle $\Gamma(X', Y')$, and it is unique.*

The case X, Y, N'. The situation if the points X, Y, N' are prescribed is quite different. Again we begin by examining an appropriate motion. It is plain in Figure 1 that the motion $Y_{60}X_{60}N'_{-120}$ is the identity, because it is a translation that fixes the point A. Consequently $N'_{120} = Y_{60}X_{60}$, and $\triangle XYN'$ is a positively oriented isosceles triangle with $\angle XN'Y = 120°$. A similar argument using the motion $Y'_{-60}X'_{-60}N_{120}$ shows that $\triangle X'Y'N$ is a negatively oriented isosceles triangle with $\angle X'NY' = -120°$.

Conversely, suppose that points P, Q, R are given (to play the role of X, Y, N'), and let A be any fixed point of the motion $Q_{60}P_{60}R_{-120}$ and $C = AQ_{60}$ and $B = CP_{60}$. Then $\triangle ACQ$ and $\triangle CBP$ are equilateral with $\angle AQC = \angle BPC = 60°$, and $\triangle BAR$ is isosceles with $\angle BRA = -120°$. But the motion $Q_{60}P_{60}R_{-120}$ is a translation, so it has a fixed point precisely when it is the identity; and it is easy to see that this occurs precisely when $\triangle PQR$ is a positively oriented isosceles triangle with $\angle PQR = 120°$. Then A can be chosen arbitrarily, and B, C determined.

Similarly, the translation $Q_{-60}P_{-60}R_{120}$ has a fixed point just when it is the identity, and this occurs precisely when $\triangle PQR$ is a negatively oriented isosceles triangle with $\angle PQR = -120°$. Again A can be chosen arbitrarily, and B, C determined: $C = AQ_{-60}$, $B = CP_{-60}$.

In either case, $\triangle ACQ$ and $\triangle CBP$ are equilateral with $\angle AQC = \angle BPC = \pm 60°$, and $\triangle BAR$ is isosceles with $\angle BRA = \pm 120°$.

When are A, B, C collinear? If A, B, C are not collinear, when are the given points P, Q, R the points X, Y, N' of $\triangle ABC$ and when are they the points X', Y', N? In other words, how is $\triangle ABC$ oriented? Here is our final converse.

Theorem 8. *Distinct points P, Q, R are given (to play the role of X, Y, N'). Let S be the point so that $\triangle PQS$ is positively oriented and equilateral, and let R_1 be the center of $\triangle PQS$ and R_2 the points symmetric to R_1 in $\overleftrightarrow{PQ}$. Let Γ_1 be the circle determined by the points Q, R_1, S and Γ_2 the circle symmetric to Γ_1 in $\overleftrightarrow{PQ}$. Then there are three points A, B, C so that $\triangle CPB$ and $\triangle AQC$ are equilateral with $\angle CPB = \angle AQC = \pm 60°$ and $\triangle RBA$ is isosceles with $\angle BRA = \pm 120°$ if and only if R is R_1 or R_2; and in either case, one of A, B, C can be chosen arbitrarily and the other two*

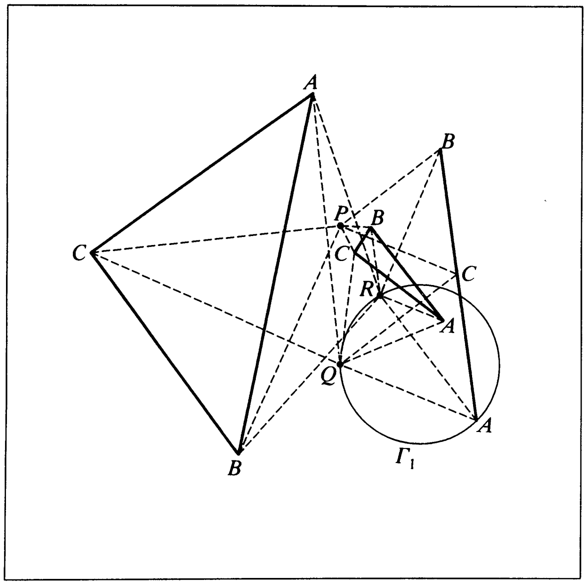

FIG. 8. The case X, Y, N'.

determined. Suppose $R = R_1$. *Then* (FIGURE 8):

(a) *If* A *lies on* Γ_1, *then* A, B, C *are collinear;*
(b) *If* A *lies inside* Γ_1, *then* $\triangle ABC$ *is positively oriented, and its points* X, Y, N' *are the given points* P, Q, R;
(c) *If* A *lies outside* Γ_1, *then* $\triangle ABC$ *is negatively oriented, and its points* X', Y', N *are the given points* P, Q, R.

If $R = R_2$, *the orientation of* $\triangle ABC$ *is reversed, but the other assertions are unchanged.*

Proof: If there are points A, B, C with the property described, the remarks prior to the statement of the theorem imply that $R = R_1$ or $R = R_2$. Suppose the former. In the complex coordinate system employed in the proof of Theorem 5 above, if A has coordinate a, then the coordinates of B and C are $b = \bar{h} - \bar{h}a$ and $c = ha$. A calculation shows that

$$(ABC) = -\frac{\sqrt{3}}{4}\left(\left|a + \frac{\sqrt{3}}{3}i\right|^2 - \frac{1}{3}\right).$$

The various claims are now immediate, and the assertions in the case $R = R_2$ follow from the symmetry in $\overleftrightarrow{PQ}$. ■

To summarize, if two different points U, V are given, let $W_1 = f_1(U, V)$ be the vertex of the positively oriented isosceles triangle with base $\overline{UV}$ and base angle 30° and $W_2 = f_2(U, V)$ the vertex of the negatively oriented isosceles triangle with base $\overline{UV}$ and base angle 30°; and let $\Gamma_j = \Gamma_j(U, V)$ be the circle tangent to $\overleftrightarrow{UV}$ at V through W_j. Then we have the following Napoleon converse:

Corollary 9. *A progenitor triangle* $\triangle ABC$ *for given points* X, Y, N' *exists precisely when* $N' = f_1(X, Y)$ *or* $N' = f_2(X, Y)$. *In the former case,* A *can be chosen* *arbitrarily inside the circle* $\Gamma_1(X, Y)$, *and* $C = AY_{60}$ *and* $B = CX_{60}$. *In the latter case,* A *can be chosen arbitrarily inside the circle* $\Gamma_2(X, Y)$, *and* $C = AY_{-60}$ *and* $B = CX_{-60}$.

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