Euclid's Ever-turning Windmill

XYZ

It will serve us in good stead with that awkward youth who is always sulkily asking us the wherefore of all these triangles, parallelograms and circles. It is no use to tell him that they are the whetstones for his wits. He is not aware that his wits need sharpening, nor would he greatly relish the prospect if he were. Indeed, he regards his discovery of the uselessness of Euclid as a proof of his already superior sharpness. So we may lawfully use lower motives with him. We may tell him that there is a Science of Trigonometry which is merely the Algebraical statement and expansion of Euclid i, 47. That it is this science which enables ships to sail in straight course, or St. Gothard Tunnels to be pierced so exactly, that engineers from Switzerland and engineers from Italy meet, within an inch or two, in the centre of the mountain after five miles of independent burrowing from opposite sides, and we can thus experto crede, *inspire the dullest with a kind of interest in his work.*

W. P. Workman, 1897 [19, p. 195]

1 Locating an unallocated lemma

Euclid (c. $\overline{325}$–c. $\overline{265}$) presents a first proof of the Pythagorean Proposition, that the square on the hypotenuse of a right triangle is equal (in area) to (the sum of) the squares on the legs, towards the end of *Book I* of his *Elements*, in Proposition *I.47* (interpolating our understanding of *area* and *sum* in Euclid's succinct formulation). Over the centuries, this proof has drawn both bouquets and brickbats, while the diagram that traditionally accompanies this proof, pored over intently by generations of students, has acquired a corresponding variety of nicknames (see the discussion in [6, Vol. 1, pp. 415, 417–418]). Of these sobriquets, that of the *Windmill* [7, Vol. 1, p. 378] does at least have the merit of reminding us of the crucial role of rotation in Euclid's manipulation of areas, that, with reference to Figure 1, the pairs of triangles $\triangle ABD$ with $\triangle KBC$ and $\triangle BAF$ with $\triangle JAC$ in play in the proof are, not just congruent, but can be obtained one from the other by quarter turns.

But Euclid's Windmill also seems to turn over in the minds of those who study it, revealing further properties and generalisations. This propensity must have set in fairly early, to mixed response, since already Proclus Diadochus (411–485) remarks on it somewhat adversely in his commentary on the *Elements* (as quoted in [6, Vol. 1, p. 366])

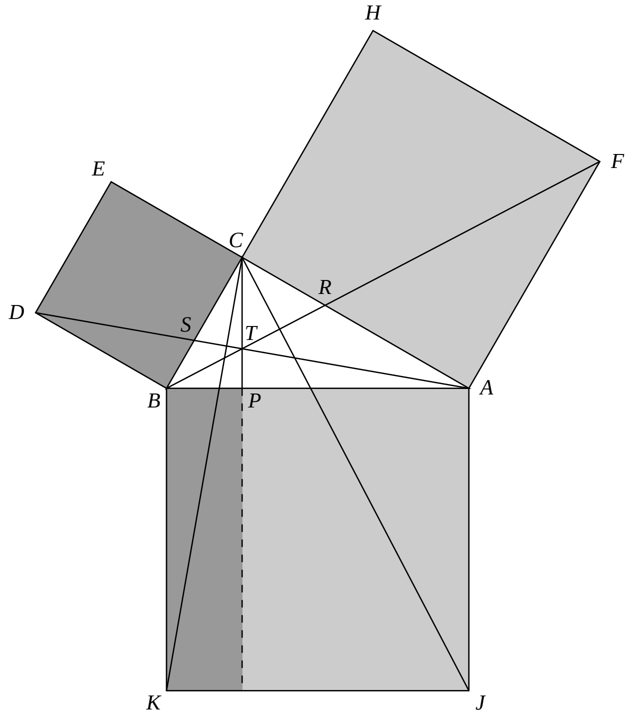
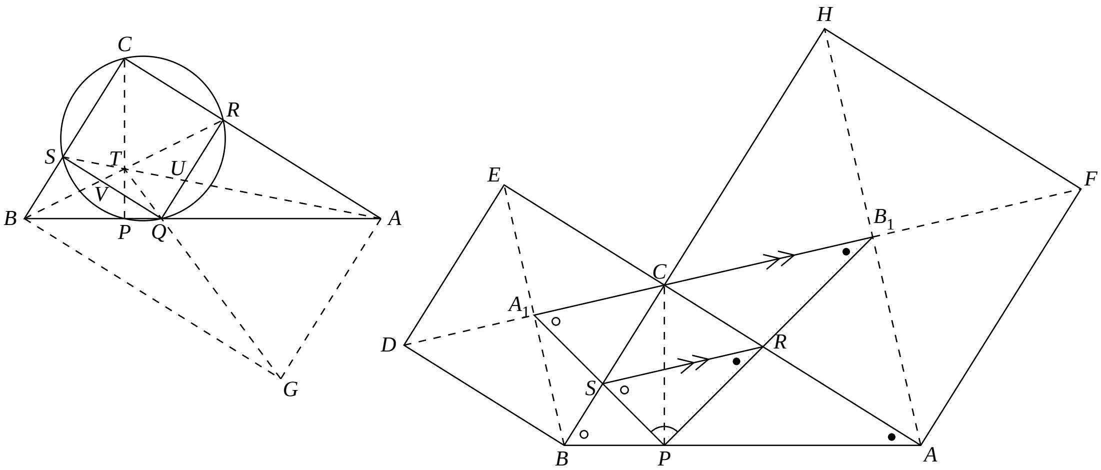


Figure 1: Euclid's Windmill

> *The demonstration* [of I.47] *by the writer of the* Elements *being clear, I consider that it is unnecessary to add anything further, and that we may be satisfied with what has been written, since in fact those who have added anything more, like Pappus and Heron, were obliged to draw upon what is proved in the sixth Book, for no really useful object.*

It is worth perhaps keeping this longer historical perspective in mind, because it seems sometimes suggested that closer investigation of Euclid's Windmill only got under way with a letter [15] from Vecten on 30 June, 1817 to Joseph Diaz Gergonne (1771–1859), the Editor of *Annales de Mathématiques pures and appliquées*, familiarly known as *Gergonne's Annales*. Writers of this later epoch were certainly not inhibited by Proclus' strictures on proof technique or utility, if indeed they knew about them.

Now, whether or not the observation is of any moment, it can hardly escape notice in Figure 1 that the lines AD and BF intersect on the line through C perpendicular to the hypotenuse AC, all the more in that this is the demarcation line between the two rectangles into which the square on the hypotenuse is divided in the course of the proof presented by Euclid (for a general reference for triangle centres, see [9]; see [17, p. 80] for a recent anthology of coincidences). Perhaps piquing curiosity serves its own purpose, given the lively and protracted comment this coincidence has excited. In contrast, much less attention has been given to the final, stray lemma offered by Pappus (c. 290–c. 350) at the end of the guided tour of the works comprising the "*Treasury of Analysis*" he provides in Book VII of the *Mathematical Collections*, to the extent that it remains unsourced and seemingly otherwise unrecognised (see [7,



(i) Theorem 1 (ii) Theorem 2

Figure 2: On location

Vol. 2, pp. 426–427]).

Lemma (Pappus) *Let $\triangle ABC$ be a right triangle with right angle at C and let AC and BC be divided at R' and S' respectively such that*

$$BS' : S'C = BC : AC = CR' : R'A. \tag{1}$$

If AS' and BR' intersect at T', then CT' is perpendicular to AB.

Clearly, the language in which this Lemma is couched has moved on from *Elements I*. However, with Figure 1 in view, there is the suspicion that the result is not unfamiliar. Indeed, as we confirm in Section 5, the Lemma is an intrinsic restatement of the coincidence of AD, BF and CP shorn of the squares on the sides of the right triangle $\triangle ABC$. So, quite possibly it preserves something of the material Proclus was inclined to reject. But we rescue the Lemma from comparative neglect in which it has languished, because, in coming to view the coincidence differently, we find an invitation to see further, as suggested in Figure 2.

In order to summarise these further findings, let $AGBC$ be a rectangle (as in Figure 2(i)), with P the foot of the perpendicular from C onto AB; Q the intersection of the angle bisector of $\angle ACB$ with AB; R and S the points of intersection of Euclid's lines BF and AD with AC and BC (as in Figure 1); and T the point of intersection of AS and BR. To identify two further points in Figure 1, let AD intersect CJ in U; and let BF intersect CK in V. Finally. let A_1 and B_1 be the centres of the squares placed externally on BC and AC (see Figure 2(ii)).

Theorem 1 (i): *T lies on CP and GQ produced;*
(ii): *$CRQS$ is a square inscribed in $\triangle ABC$;*
(iii): *P lies on the circumcircle of this inscribed square; and*
(iv): *U is on QR; V is on QS; UV is parallel to AB; and tirangles $\triangle VUQ$ and $\triangle ABC$ are similar.*

Theorem 2 (i): *A_1SP and B_1RP are straight lines bisecting the angles $\angle APC$ and $\angle BPC$ respectively; and*
(ii): *the triangles $\triangle B_1A_1P$ and $\triangle RSP$ are similar to $\triangle ABC$, with CP the angle bisector of $\angle A_1PB_1$.*

Some modern Proclus might contend that these Theorems do not serve any "*really useful object*" when weighed in the balance against the Pythagorean Proposition. Naturally, few results would count by that measure. Rather, our interest lies in fostering the habits of noticing and proving, without which appreciation of the great theorems of mathematics would become yet more difficult. In this regard, it is always curious what is recorded and what seems to be overlooked. For example, the equality of the intercepts CR and CS in Figure 1 is sometimes noticed along with the coincidence of the lines AD, BF and CP at T (see, for example, [1, Prop. 23, pp. 16–17]). However, when it comes to discussion of inscribed squares, the connection with Figure 1 goes unmentioned (compare [1, 8, 18]).

As it happens, not only was this coincidence of lines in Euclid's Windmill known to Heron (c. 10–c. 75), but also, in a remarkable survival, a proof has come down to us from Heron's commentaries framed in terms of *Elements 1*, suggesting that early writers were not so indifferent to proof techniques as Proclus asserts, and answering into the bargain a question [5] raised anew in 1823 by J. Hamett in *Philosophical Magazine*. For, Heron calls on *Elements I.43*, that in any parallelogram, the complements of parallelograms about a common diagonal are equal in area — in effect, working this proposition backwards and forwards avoids the need to appeal the proportionality of similar triangles. This primitive approach makes a good starting point as we build towards a proof of our Theorems. So, in Section 2, we explore what information can be gleaned this way, before probing further in Sections 3–6 by means of similar triangles and cyclic quadrilaterals, returning in Section 7, our final section, to provide an historical retrospective.

2 Exercises on *Elements I.43* and its converse

At first glance, *Elements I.43* might seem unprepossessing, even inconsequential: the statement is opaque; the substance elementary. In Figure 3(i), $ABDC$, $AGXE$ and $DIXF$ are the parallelograms about a common diagonal while $CEXI$ and $BFXG$ are the two complements in question. The common diagonal splits the parallelograms into three pairs of congruent triangles, $\triangle ABD$ with $\triangle DCA$, $\triangle AGX$ with $\triangle XEA$ and $\triangle XFD$ with $\triangle DIX$. Hence, the complements $BFXG$ and $CEXI$ are equal in area since they can be obtained by starting from congruent triangles and excising congruent triangles:

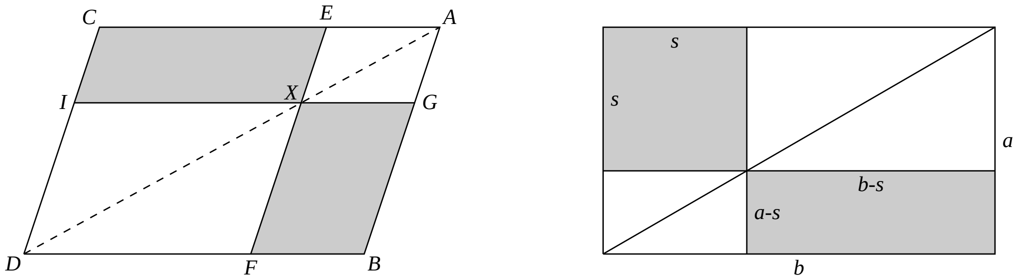
$$BFXG = \triangle ABD - \triangle AGX - \triangle XFD.$$

and

$$CEXI = \triangle DCA - \triangle XEA - \triangle DIX.$$

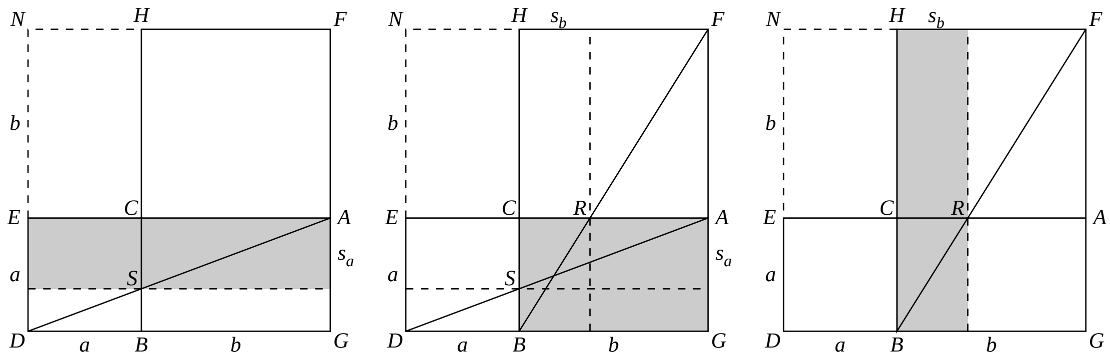
Euclid does not prove a converse to *I.43*. But it is straightforward to reverse the foregoing argument to conclude that, if $ABDC$ is a parallelogram divided by lines EF and GI parallel to AB and AC respectively so that $BFXG$ and $CEXI$ are equal in area, then X is on the diagonal AD (refer again to Figure 3(i)). For, under these hypotheses, the regions

$$AGBFDX = \triangle AGX + \triangle XFD + BFXG$$



(i) (ii)

Figure 3: *Elements I.43*



(i) (ii) (iii)

Figure 4: Exercise on *I.43* — CR = CS

and

$$DICEAX = \triangle XEA + \triangle DIX + CEXI$$

partition the parallelogram $ABDC$ and are equal in area. Consequently these regions must both occupy half the area of the parallelogram. But this is also the case for the triangles $\triangle ABD$ and $\triangle DCA$. So, $\triangle AXD$ is a triangle with no area, implying that X is on AD.

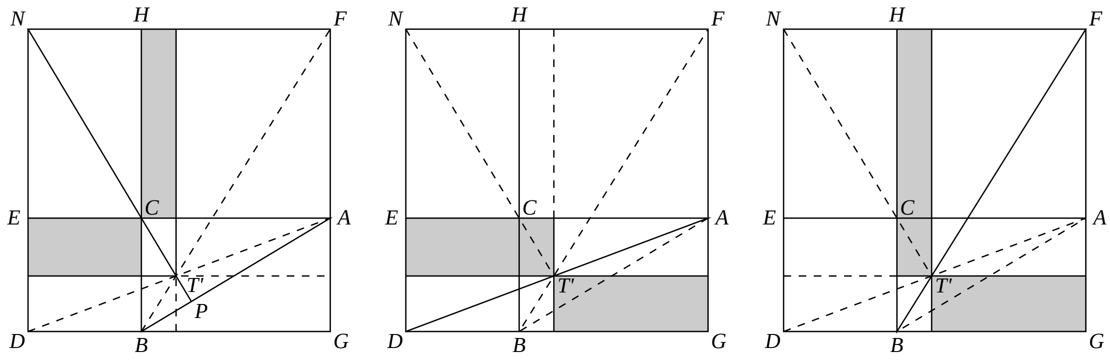
As a first application of *I.43*, consider a rectangle with sides a and b as in Figure 3(ii) where a square of side w has been inscribed between a corner and a diagonal. By *I.43*, the shaded rectangle in Figure 3(ii) inscribed between the opposite corner and the diagonal has the same area as the inscribed square. So, replacing this shaded rectangle by a second copy of the inscribed square, we may reorganise the original rectangle into one of the same area having sides w and $a + b$, showing that

$$w(a + b) = ab \tag{2}$$

But, even without recourse to this reorganisation of areas, (2) is implied algebraically by the balance between the shaded regions in Figure 3(ii):

$$w^2 = (a - w)(b - w).$$

Either way, in a right triangle with legs a and b, a square inscribed so as to have sides on those legs has side $w = ab/(a + b)$ (compare [18]).



(i) (ii) (iii)

Figure 5: Exercise on *I.43* — a coincidence

Now, let us turn to the observation that, in Figure 1, $CR = CS$. In Figure 4(i), the shaded rectangle is cut out of the rectangle $AGDE$ by the line through S parallel to AE, so that S is the intersection of this line with the diagonal AD. So, by *I.43*, the shaded region in Figure 4(i) outside the rectangle $AGBC$ is equal in area to the unshaded portion of this rectangle, which is to say that the shaded areas in Figures 4(i) and (ii) are equal. Similarly, applying *I.43* to the rectangle $BHFG$ shows that the shaded areas in Figures 4(ii) and (iii) are also equal. Hence, in the notation of Figure 4,

$$s_a(a + b) = ab = s_b(a + b), \tag{3}$$

and so $CR = s_b = s_a = CS$. Moreover, comparison of (2) and (3) reveals that C, R and S are vertices of a square inscribed in $\triangle ABC$ with the lines through R and S parallel to BC and AC meeting on the hypotenuse AB at the fourth vertex of the square, namely Q.

But we can also establish this coincidence at Q by way of illustrating the use of the converse of *Elements I.43*. For, the shaded area in Figure 4(i) *outside* the rectangle $AGBC$ is equal in area to the shaded area *inside* this rectangle when it comes to Figure 4(iii). So, from our previous argument, the *unshaded* area inside the rectangle $AGBC$ in Figure 4(i) and the *shaded* area inside it in Figure 4(iii) are equal. Deleting the region common to both areas, we have balancing rectangles set into opposite corners, C and G, of the rectangle $AGBC$ cut off by the lines through R and S parallel to BC and AC respectively (compare the move from Figures 6(i) and (ii) to Figure 6(iii)). Thus, by the converse to *I.43*, these lines meet on the diagonal AB of the rectangle $AGBC$. Since we also know that the balancing rectangle set into the corner C is in fact a square, this point of intersection is Q.

These preliminary skirmishes with *Elements I.43* and its converse help prepare us for Heron's demonstration that the lines AD, BF and CP in Euclid's Windmill are coincident. Heron supposes that AD and CP meet at, say, T' and aims to show that T' is also on BF. First of all, he checks that, if DE and HF meet at N, as in Figure 5, then PC produced passes through N. This allows him to use *I.43* to infer that complements about $T'CN$ shown shaded in Figure 5(i) are equal in area. Next attention shifts to the rectangle $AGDE$, where another instance of *I.43* shows that the complements about $AT'D$ shown shaded in Figure 5(ii) also have

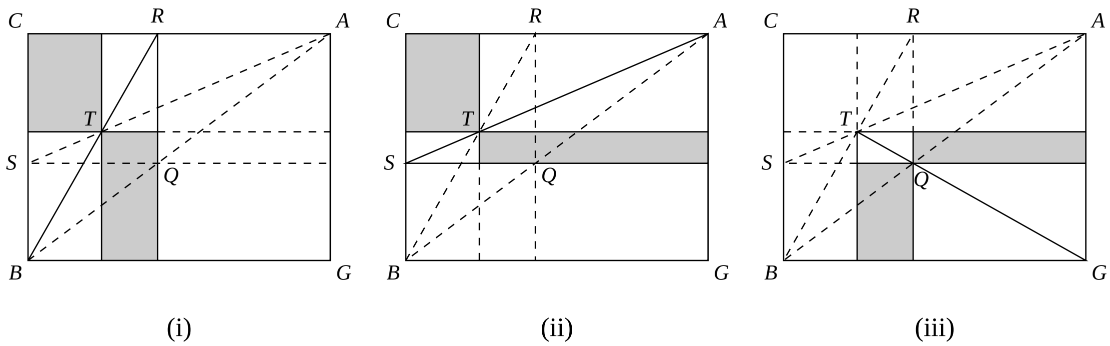


Figure 6: Exercise on *I.43* — another coincidence

equal area. Combining this information, Heron finds balancing rectangles set into opposite corners, G and H, of the rectangle $BHFG$ and having vertex T' in common. Hence, by the converse to *I.43*, T' lies on the diagonal BF of the containing rectangle $BHFG$, confirming the coincidence observed in Figure 1 at T.

In working through the demonstrations illustrated in Figures 4 and 5, it is natural to wonder how the points of coincidence Q and T that play pivotal rôles in them might be related? Here we can learn from Heron's example. By applying *Elements I.43* to rectangles with BR and AS as diagonals, the complements shown shaded in Figure 6(i) and (ii) are equal in area. Combining this information, the shaded rectangles in Figure 6(iii) with common vertex at Q also have equal area. So, the converse of *I.43* implies that G, Q and T are collinear.

3 More than a coincidence

Euclid does envisage squares on the sides of general triangles, not just right triangles. He gives a version of the *Law of Cosines*, generalising the Pythagorean Proposition, in two propositions late in *Elements II* — *II.12* for obtuse angles and *II.13* for acute angles, with *Data 63* providing a complement to them. For that matter, *Elements III.36* gives yet another generalisation of *I.47*, in effect anticipating one ascribed to Thabit ibn Qurra al-Harrani (836–901), although by that stage of the *Elements* Euclid 's focus is on the geometry of the circle, leaving the explicit manipulation of areas behind.

Interestingly enough, the proof of *I.47* captured in Euclid's Windmill carries over to *II.12, 13*, but that is not how Euclid proved them, nor has this line of argument been detected prior to 1647 (see [6, Vol. 1, p. 404]). Indeed, infrequent adoption since then lends this style of proof almost an air of novelty when it does appear. Vecten's letter [15] in 1817 was a departure from this apparent incuriosity in scrutinising the configuration of squares placed externally on the sides of a general triangle, prompted by a search for a proof of the continued coincidence of lines observed in Euclid's Windmill.

But this general concurrency is not difficult to prove. A proof by similar triangles is suggested in Figure 7(i) by framing each of the squares on the sides AC and BC with its own set of four congruent right triangles. If, AD meets CP in T', then the

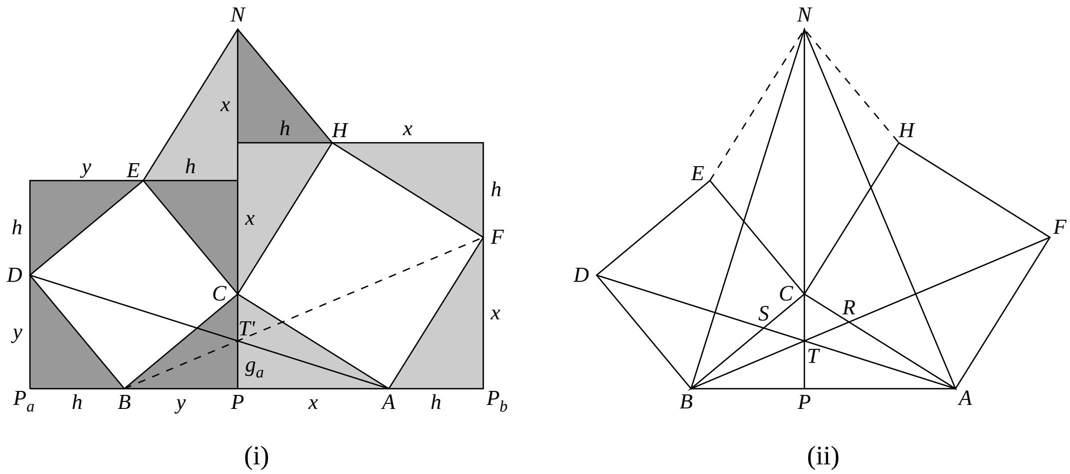


Figure 7: A general coincidence

triangles $\triangle AP_aD$ and $\triangle APT'$ are similar. So,

$$PT' : AP = P_aD : AP_a,$$

or, in the notation of Figure 7(i), with $g_a = PT'$ the intercept of AD on PC:

$$\frac{g_a}{x} = \frac{y}{h+x+y}.$$

Hence,

$$g_a = \frac{xy}{h+x+y}. \tag{4}$$

Since the right-hand side of (4) is symmetric in x and y, the intercept g_b of BF on CP will be given by the same expression, that is, $g_b = g_a$ and the lines AD, BF and CP are coincident.

However a demonstration more in the spirit of Euclid's Windmill, and in essence drawing only on *Elements I*, can be given as follows (the case of a right triangle was already treated in much this way in [4, (b)]; compare also [16]). Consider Figure 1 again, but now without the assumption that $\triangle ABC$ is a right triangle. We rotate the triangles $\triangle ACJ$ and $\triangle BCK$ a half-turn about the mid-points of AC and BC respectively to create the triangle $\triangle ABN$ shown in Figure 7(ii). Clearly $\triangle ABN$ is congruent to $\triangle JKC$, indeed a translation of it in the direction PC. Notice also that $\triangle CNH$ and $\triangle NCE$ are both congruent to $\triangle ABC$, so that $CENH$ is a parallelogram and we recover the upper left-hand portions in Figure 4 and 5 on reverting to the case where $\triangle ABC$ has a right angle at C. Thus, NA and NB are parallel to CJ and CK, which are related to BF and AD respectively by quarter turns, as already noted in our introduction to the Windmill, while NCP is a straight line — a point Heron also had to check in the more restricted circumstances of Figure 5, as noted in Section 2.

We now see AD, BF and CP afresh as the altitudes of $\triangle ABN$, so their concurrency follows from the general concurrency of the altitudes of a triangle. This latter concurrency is often proved by means of the circle geometry of *Elements III*. But this can be avoided by deducing the result from prior knowledge that the perpendicular bisectors of the sides of a triangle are coincident at the circumcentre of the triangle (compare [17, pp. 128–129]). It is true that Euclid relegates the construction of the circumcircle of a triangle to *Elements IV*, in *IV5*, along with other constructions for a range of circumscribed and inscribed figures. But Euclid's proof of *IV.5* appeals directly to *Elements I*, making for a simpler demonstration of the concurrency of the altitudes than proceeding *via* circle geometry.

Looking back to Figure 1, but still without restriction on $\triangle ABC$, we can now turn the argument around to extract further information. To this end, recall that AD and CJ intersect in U and that BF and CK intersect in V (see Figure 1). Then AD and BF are altitudes of the new triangle $\triangle UVC$ and intersect at T. Since T is on CP, it follows that CP is the third altitude of $\triangle UVC$ and hence that UV is parallel to AB. Of course, it is possible to give a direct proof that UV and AB are parallel, for example by similar triangles, and so to deduce that AD, BF and CP are coincident from the concurrency of the altitudes of $\triangle UVC$. For the record, the perpendicular distance alike of U or V from AB is, in the notation of Figure 7(i)

$$\frac{xy(x+y)}{(h+x+y)^2 - xy},$$

which reduces to $h^2/(2h+x+y)$ when $\triangle ABC$ has a right angle at C since then $h^2 = xy$.

The three line segments of this kind, produced as necessary, will intersect to give a triangle similar to $\triangle ABC$. In the case where $\triangle ABC$ has a right angle at C, the other two segments besides UV are, in fact, the intercepts CR and CS, making it easy to identify the triangle they define. An an agreeable bonus, U and V are then on the sides QR and QS of the inscribed square $CRQS$ identified in Section 2. This follows on showing, either as another exercise on the lines of Secetion 2 or by further arguments using similar triangles, that the perpendicular distances of U from BC and V from AC is the same as the side s of the square $CRQS$ inscribded in $\triangle ABC$ (compare (2)). Since we now have both that UV is parallel to AB and that $\angle UQV = \angle RQS = \pi/2$, triangles $\triangle VUQ$ and $\triangle ABC$ are similar.

4 The inscribed square

We have seen in Section 2 that *Elements I.43* and its converse are sufficient to identify $CRQS$ as a square inscribed in the right triangle $\triangle ABC$. Alternatively, without imposing the limitations of *Elements I*, we can rework the proof that the intercepts CR and CS in Figure 1 are equal in terms of similar triangles. In Figure 4(i), the triangles $\triangle ADE$ and $\triangle ACS$ are similar, so that, in the notation there,

$$\frac{s_a}{b} = \frac{a}{a+b},$$

or

$$s_a = \frac{ab}{a+b}. \tag{5}$$

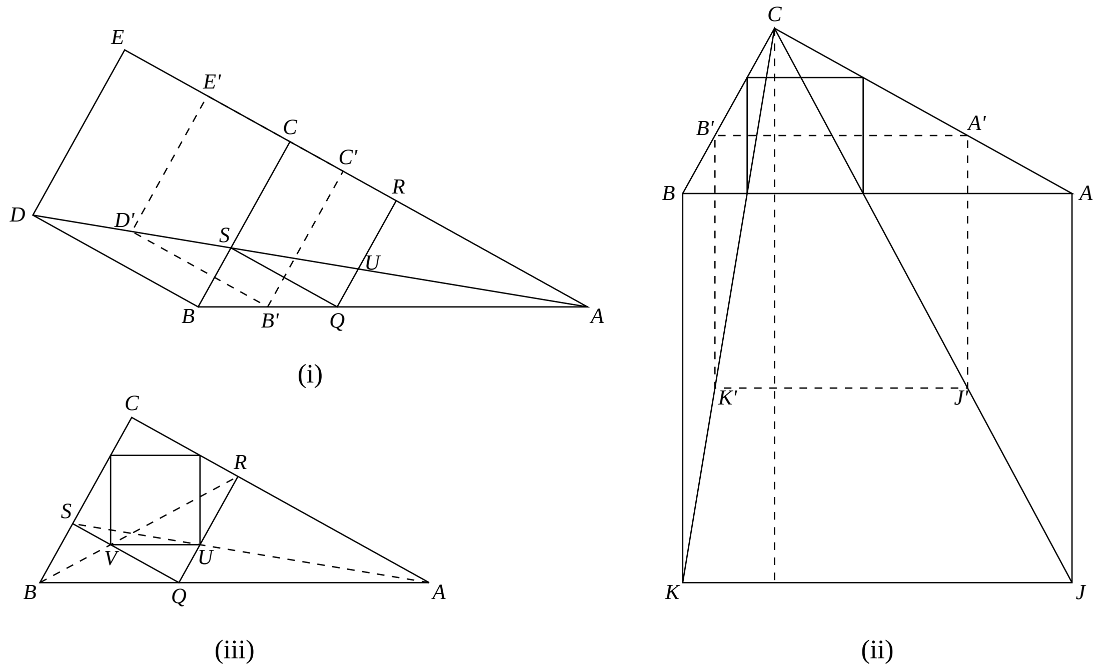


Figure 8: In perspective

As observed with (4), the right-hand side of (5) is symmetric in a and b, indicating that, when we come to consider the similar triangles, $\triangle BFH$ and $\triangle BRC$, s_b will be given by the same expression. Consequently $CS = s_a = s_b = CR$.

The extension of this argument by similar triangles to a general triangle $\triangle ABC$ reveals that, if the intercepts CR and CS in Figure 7(ii) are equal then $\triangle ABC$ is either isosceles, with $AC = BC$, or the angle at C is right. To be more specific, if $\angle ACB = \pi - \theta$, then it can be shown that these intercepts in Figure 7(ii) are given by

$$CR = \frac{ab(\sin\theta - \cos\theta)}{a + b\sin\theta}; \quad CS = \frac{ab(\sin\theta - \cos\theta)}{b + a\sin\theta}.$$

Now, a rich source of geometrically similar figures is provided by perspective. As Figure 8(i) suggests, for a right triangle $\triangle ABC$ with right angle at C, there will be an inscribed square in perspective from A with square placed externally on BC. But such an inscribed square will then also be in perspective from B with the square placed externally on AC. These perspectives give another way of seeing that the intercepts CR and CS in Figure 1 are equal and so of identifying $CRQS$ as a square inscribed in $\triangle ABC$.

A triangle admits an inscribed square standing on a side of the triangle if the altitude perpendicular to that side is internal. In general, then, a triangle has either three or one inscribed squares, with right triangles a borderline case in which two of the altitudes are sides. A standard construction of an inscribed square is by perspective from a square placed either on an internal altitude or on the side of the triangle perpendicular to that altitude (see, for example, [18, (a)]; a rather different construction is presented in [1, Prop, 14, pp. 10–11]). While Figure 8(i) is an instance of this construction, Figure 8(ii) shows how another part of Euclid's Windmill is naturally associated with the construction of an inscribed square standing on the hypotenuse of a right triangle. By the same token, there will also be a square on

UV in perspective from C with the square placed externally on AB. In the case where $\triangle ABC$ has a right angle at C, this latter square is inscribed in $CRQS$, as in Figure 8(iii).

5 The unallocated lemma

First of all, let us identify the points of intersection R and S in Figure 1 with the dividing points R' and S' specified in Pappus' unallocated lemma by (1). In Figure 1, the triangles $\triangle AFR$ and $\triangle BCR$ are similar, while $AFHC$ is a square. So,

$$CR : RA = BC : AF = BC : AC.$$

Similarly, considering the similar triangles $\triangle ACS$ and $\triangle BDS$ together with the square $BDEC$,

$$BS : SC = BD : AC = BC : AC.$$

Hence (1) holds for $R' = R$ and $S' = S$. Conversely, if (1) holds, we can argue that $AS'D$ and $BR'F$ are straight lines, showing that R' and S' are the points of intersection R and S respectively. With this identification in hand, we now drop the primes, and work only with R and S, so that now (1) reads

$$BS : SC = BC : AC = CR : RA. \tag{6}$$

Thus, we can now complete the proof of the lemma by appeal to our discussion in Sections 2 and 3.

However, we are hampered in working through the proof of this unallocated lemma given in the final portion of *Mathematical Collections VII* because the received text is garbled. The proof starts off by showing that (6) implies that the intercepts CR and CS are equal. But then an impatient interpolator interrupts to point out that, if we have the altitude CP of the triangle in addition to (6), then we can infer that

- **(a):** R and S are on the angle bisectors of the angles $\angle APC$ and $\angle BPC$ respectively;
- **(b):** $\angle RPS$ is right, so that $CRPS$ is a cyclic quadrilateral; and
- **(c):** consequently CR and CS are equal, since they subtend equal angles at P.

Curiously enough, neither line of attack seems to get to grips with the lemma as stated.

We return to consider (b) in next section. But (a) is promising in its own right. It is a matter of recognising in (6) a hint of the characterisation of angle bisectors by means of ratios featured in *Elements VI.3*. For, if P is the foot of the perpendicular from C onto AB, then all three right triangles $\triangle ABC, \triangle ACP$ and $\triangle CBP$ are similar, so that

$$CP : AP = BC : AC = BP : CP. \tag{7}$$

Comparing (7) with (6) yields

$$CR : AR = CP : AP; \quad BS : CS = BP : CP.$$

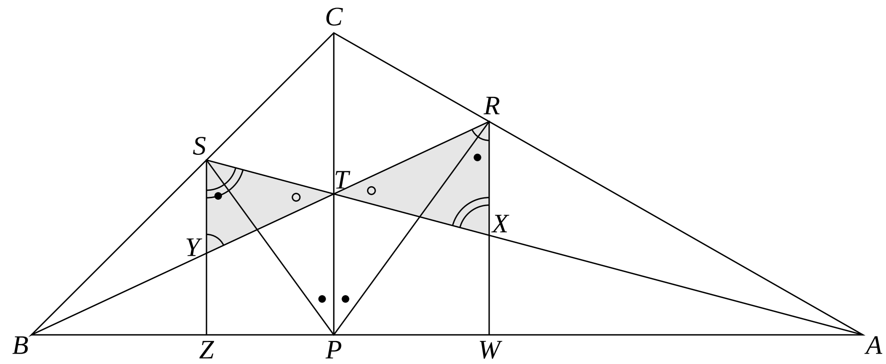


Figure 9: Shared and inverted perspective

But the equality of these ratios is precisely the condition in *Elements VI.3* ensuring that R and S lie on the angle bisectors of $\angle APC$ and $\angle BPC$ respectively.

Since these latter angles are right, all the angles $\angle APR, \angle CPR, \angle CPS$ and $\angle BPS$ are equal, in fact all are $\pi/4$. Thus, with angle bisectors in mind, CP is also the angle bisector of $\angle RPS$. Now, there is a general lemma about the altitude of a triangle as an angle bisector that is sometimes mentioned in connection with the *orthic* triangle formed by joining the feet of the altitudes (see, for example, [12, §6.2, p. 11]. In our present circumstances, we can apply this general lemma to infer Pappus' unallocated lemma from (a) in a comparatively self-contained manner.

Lemma *Let $\triangle ABC$ be an arbitrary triangle; let R and S be points on AC and BC respectively with T the point of intersection of AS and BR; and let P be the foot of the perpendicular from C onto AB.*

(i): *If T lies on CP, then CP is the angle bisector of $\angle RPS$; and conversely*

(ii): *If CP is the angle bisector of $\angle RPS$, then T lies on CP.*

Proof of Lemma (i): Suppose that, as indicated in Figure 9, W and Z are the feet of the perpendiculars from R and S respectively and that AS and BR intersect RW and SZ in X and Y respectively. Thus, CP is shared in perspective from both A and B (compare the perspectival proof of Napoleon's Theorem in [14, (b)]). In particular, triangles $\triangle ACP$ and $\triangle ARW$ are in perspective from A, so that

$$RX : RW = CT : CP.$$

But similarly, looking from B, triangles $\triangle BCP$ and $\triangle BSZ$ are in perspective, giving

$$SY : SZ = CT : CP.$$

Combining these last two equalities and rearranging the ratios yields

$$SZ : RW = SY : RX. \tag{8}$$

Since RX and SY are parallel, the triangles $\triangle TRX$ and $\triangle TYS$, shown shaded in Figure 9, are in inverted perspective from T and so similar. The heights of this pair of triangles must therefore stand in the same proportion as their bases. With respect to their parallel sides as bases, this means that

$$ZP : WP = SY : RW. \qquad (9)$$

Switching attention now to the right triangles $\triangle RPW$ and $\triangle SPZ$, combination of (8) and (9) reveals that they have their legs in the same proportion:

$$ZP : SZ = WP : RW.$$

This implies that these right triangles are similar. It follows that

$$\angle CPS = \angle PSZ = \angle PRW = \angle CPR,$$

bearing in mind that RW and SZ are parallel to CP. This completes our proof that CP is the angle bisector of $\angle RPS$.

Proof of Lemma (ii): Conversely, suppose that AS and CP intersect at T' and that BT' intersects AC at R'. Turning this around, AS and BR' intersect at T' on CP, so part (i) of the Lemma applies and this CP is the angle bisector of $\angle R'PS$. But if CP is known to be the angle bisector of $\angle RPS$, then $R' = R$ and so $T' = T$. We conclude that T, the intersection of AS and BR, lies on CP as claimed.

Proof of Pappus' Lemma: In leading up to the statement of the foregoing Lemma, we saw that CP is the angle bisector of $\angle RPS$ when R and S are points on AC and BC satisfying (6). So, Pappus' unallocated Lemma now follows as an immediate consequence of part (ii) of our Lemma.

The rehearsal of these proofs allows us to reflect on the information placed at our disposal by the hypothesis of Pappus' Lemma in the context of what we know about right triangles. The formulation of Pappus' Lemma is akin to that of a celebrated general theorem on concurrency of lines in a general triangle $\triangle ABC$: if P', S' and R' are points on the sides AB, DC and CA respectively, then AS', BR' and CS' are concurrent if and only if

$$\frac{AP'}{P'B}.\frac{BS'}{S'C}.\frac{CR'}{R'A} = 1.$$

The hypothesis (i) is exactly matched by the fact that, for a *right* triangle $\triangle ABC$ with right angle at C, P' is the foot of the altitude at C when it divides the hypotenuse in duplicate ratio to the legs:

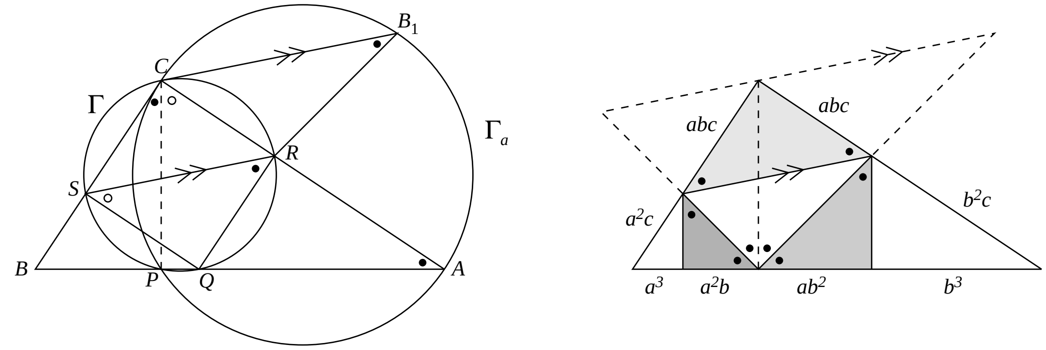
$$AP' : P'B = AC^2 : BC^2.$$

Appeal to this general result therefore clinches swift confirmation of Pappus' Lemma — this is, in effect, the opening line of attack that Gergonne mounts in [4, (a)]) in rejoinder to [5]. Unfortunately, such reasoning would seem anachronistic, since Giovanni Ceva (1647–1734) only published his theorem in 1678. However, it does enable us to extract a characterisation of right triangles.

Theorem (Pappus-Ceva) *Suppose that the sides AC and BC of a triangle $\triangle ABC$ are divided at R' and S' respectively such that*

$$BS' : S'C = BC : AC = CR' : R'A.$$

Let AS' and BR' intersect at T'. Then $\triangle ABC$ is a right triangle with right angle at C if and only if CT' is perpendicular to AB.



(i) (ii)

Figure 10: Cyclic quadrilaterals and similar triangles

6 Cyclic quadrilaterals and similar triangles

Let us now return to pick up observation (b) in the previous Section that $CPRS$ is a cyclic quadrilateral. Since $\angle RPS$, as well as $\angle RCS$ and $\angle RQS$, is right, we have a *five-point* circle Γ through C, R, Q, P and S with diameter RS (see Figure 10(i)). Thus, we have now gathered proofs of all the assertions in Theorem 1 having already dealt with parts (i) and (ii) in Sections 2 and 4. Having just mentioned Ceva's theorem at the end of the previous section, we might add that it can be used to establish a more general result on concurrencies associated with a *six-point* circle (see [17, pp. 104, 144–145]) Suppose a circle intersects each side of $\triangle ABC$ in two points, say, P', P'' on AB, S', S'' on BC and R', R'' on CA, by extension of our previous notation. Then, AS', BR' and CP' are coincident if and only if AS'', BR'' and CP'' are coincident. Our five-point circle may be viewed as a limiting case.

Turning to Theorem 2, we also see that $\angle PRS$ and $\angle PCS$ are equal, being angles subtended by PS on the same arc of the circle Γ. But $\triangle ABC$ and $\triangle CBP$ are similar, so $\angle PCS$, that is $\angle PCB$, is equal in turn to $\angle BAC$. Similarly, $\angle PSR$ and $\angle ABC$ are equal. It follows that $\triangle RSP$ and $\triangle ABC$ are similar right triangles.

On introducing B_1, the centre of the square placed externally on AC, we see that, as shown in Figure 10(ii), there is a further circle, Γ_a, say, on AC as diameter and passing through P and B_1, since $\angle APC$ and AB_1C are both right. But as the chords $AB1$ and CB_1 of Γ_a are equal, they subtend equal angles at P, that is to say that PB_1 is the angle bisector of $\angle APC$. Our discussion of observation (a) in the previous Section now places R on B_1P. Moreover, $\angle CB_1P$ is equal to $\angle CAP$, that is, $\angle CAB$, as both are subtended by the chord CP on the same arc of Γ_a.

An exactly similar argument shows that for A_1, the centre of the square placed externally on BC, A_1SP is a straight line bisecting $\angle BPC$ while $\angle CA_1P$ is equal to $\angle CBA$. But B_1CA_1 is a straight line, since

$$\angle UCV = \angle B_1CA + \angle ACB + \angle BCA_1 = \pi/4 + \pi/2 + \pi/4 = \pi.$$

Hence, $\triangle B_1A_1P$ and $\triangle ABC$ are also similar right triangles. Taking the last three paragraphs together, we see that the proof of Theorem 2 is complete.

Notice that the proportions in which the similar right triangle $\triangle RPS, \triangle B_1PA_1$ and $\triangle ABC$ stand to one another can be obtained by computing the hypotenuse of each in terms of the legs $a = AC$ and $b = BC$ of $\triangle ABC$:

$$\frac{ab\sqrt{2}}{a+b} : \frac{a+b}{\sqrt{2}} : \sqrt{a^2+b^2}.$$

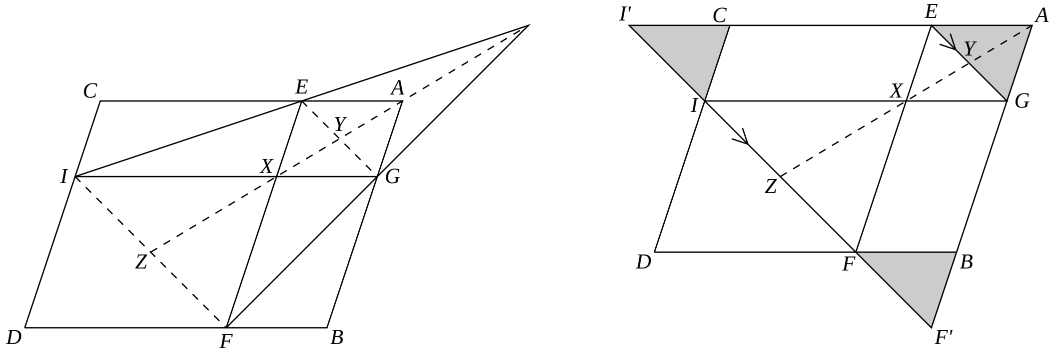
The presence of the three isosceles right triangles shown shaded in Figure 10(ii) facilitates the calculation of several other lengths in $\triangle ABC$ — a scaling factor of $(a+b)\sqrt{a^2+b^2} = (a+b)c$ has been applied in Figure 10(ii).

7 Retrospective

In Euclid's development of *Elements I*, *I.43* is the foundation of what is termed "*the application of area*", beginning in the very next proposition, *I.44*, on the construction of a parallelogram having the same angle and area as a given parallelogram but now with one side prescribed. Despite appearing to give something for nothing, this method is sufficient for the solution of quadratic equations stated in terms of length and areas. Proclus was so taken with *I.44* as to attribute it to "godlike men of old", while Thomas Little Heath (1861–1940) wrote in endorsement that it "will always remain one of the most impressive in all geometry" (see [6, Vol. 1, pp. 342–345] or [7, Vol. 1, pp. 150–154]; and compare [2, pp. 34–38]). However, the use that Heron makes of *I.43* and its converse in establishing concurrency is rather different, as well as less well-known — it passes unnoticed in [2]. (Reasoning in the style of Figure 3(i) also provides one means of deriving the heights of distant or inaccessible landmarks, but such practical problems of surveying seem more part of the ancient traditions of Chinese or Indian geometry than that of the Greeks.)

In default of access to Heron's own commentary in Greek, we are indebted to Abu'l Abbas al-Fadl ibn Hatim al-Nayrizi (c. 875–c. 940) for thinking to acquaint readers of Heron's proof in a commentary in Arabic on *Elements I* (the commentary is available in English translation in [11]). The proof clearly made a good impression on Heath as he describes it as a "*tour de force*" in his edition of Euclid's *Elements* [6, Vol. 1, p. 367] and "*worthy of note*" in his *History of Greek Mathematics* [7, Vol. 2, p. 426], including it at some length in both. But, at least as Heath presents the proof, it may yet be more instructive than even he allows, in that it has a curious gap: Heron's preparatory lemma does not apply directly to the figure under discussion, in effect Figure 3(i).

Whereas, in Section 2, we demonstrated the converse of *Elements I.43* by reversing Euclid's proof for *I.43*, for some reason Heron tempts logic by essaying a more complicated line of argument. He starts out by showing, perfectly correctly, that triangles in perspective from a common vertex have a common median through that vertex, that is, that the mid-points of sides opposite the common vertex are also in perspective from it. At several places earlier we have encountered figures in perspective, but this result on medians is within the scope of *Elements I*. Next comes some checking that the conditions in the preparatory lemma hold in Figure 3(i), that AX bisects EG and that the equality of $BFXG$ and $CEXI$ implies that



(i) (ii)

Figure 11: Heron's proof

IF is parallel to EG. The intent is to appeal to a shared median to infer that if AX produced meets IF in Z, then Z is mid-point of IF, from which it can be deduced that ZB is in line with AZ, as required. But where in Figure 3(i) are the triangles in perspective from a common vertex?

Perhaps the presumption is that FG and IE when produced intersect on XA produced, as suggested in Figure 11(i), since this immediately yields triangles in perspective from that point of coincidence, consistent with Heron's preparatory efforts. But this would leave us in the unsatisfactory position that elaborate edifice Heron erects to prove that one set of three lines is coincident is based on the prior concurrency of three other lines, as yet to be confirmed. Besides, in the case where $ABDC$ is a rhombus quartered by EF and GI, the lines FG, IE and XA are parallel.

As a way around this projective entanglement, we might instead produce IF in both directions to meet AB and AC in F' and I' respectively, as shown in Figure 11(ii). Now, triangles $\triangle AEG$ and $\triangle AI'F'$ are indeed in perspective from their common vertex A in view of Heron's verification that EG and IF are parallel. Thus, Heron's lemma on shared medians allows us to conclude that, as AX bisects EG in Y, AX produced bisects $I'F'$ in Z. This is not quite enough to round off Heron's proof as envisaged, since we need to show that Z bisects IF, not just $I'F'$. But triangles $\triangle BFF'$ and $\triangle CI'I$ are each congruent with $\triangle AEG$ and so congruent with each other. In particular, $FF' = EG = I'I$. Hence,

$$ZF = ZF' - FF' = I'Z - I'I = IZ,$$

confirming that Z is the mid-point of IF, as desired.

If much about Heron's proof remains enigmatic, some mystery also attaches to why the subject of Euclid's Windmill surfaced again in print as a topic for investigation so much later, with Vecten's letter [15] in 1817 and then Hamett's question [5] in 1823. Curiously enough, although Gergonne had published an extract from Vecten's letter in *Annales*, he does not mention Vecten in his response [4, (a)] to Hamett some six years later, and neither do the other contributors, B. D. C. and Paul Jean Joseph Querret (1783–1839), who joined the ensuing discussion in *Annales* [4, (b,c)]. Moreover, it was only with Querret's treatment of the problem that this exchanged regained the degree of generality already considered by Vecten — in fact, Querret goes somewhat further in taking a general triangle with similar rectangles placed externally on the edges (compare [17, p. 81]). Gergonne, does concede that his own first attempt, similar to the appeal to Ceva's Theorem at the end of Section 5, would not satisfy Hamett — or Proclus, too, for that matter. But he puts a brave face on it, stoutly defending a robust attitude to admissible methods of proof [4, (a) pp. 335–336]:

> *Cette démonstration, quelque simple et rigoureuse qu'elle soit, pourra fort bien ne pas complètement remplir l'attente de M. Hamett, qui désire qu'on ne s'y appuie sur aucune proposition postérieure à la XLVII.^e d'Euclide; mais il y en a dans Euclide, avant celle-là, beaucoup plus qu'il n'en faut pour démontrer les propriétés des triangles semblables, desquelles on déduit ensuite immédiatement le théorème sur lequel nous nous sommes appuyés. Il n'y a donc point de cercle vicieux dans tout ceci, et il ne s'agira que de disposer les propositions d'Euclide dans un ordre un peu différent; ce qu'on peut sans doute se permettre sans se rendre coupable de sacilége.*

Vecten's observations [15] on Euclid's Windmill went much further than the coincidence at T seen in Figure 1. For, whether or not $\triangle ABC$ is a right triangle, the triangles $\triangle ABF$ and $\triangle CAJ$ associated with squares placed externally on the sides of $\triangle ABC$ are related by a quarter turn. In consequence, A is equidistant from the lines BF and CJ which intersect at right angles. If this point of intersection is A_0, then A is on the angle bisector of $\angle FA_0J$. Vecten found that, with B_0 and C_0 defined analogously, the lines AA_0, BB_0 and CC_0 are also concurrent — not surprisingly, their point of coincidence came to be known as the *Vecten's Point* of $\triangle ABC$. Thus, Vecten's Point was suggested by consideration of Figure 1, but, unlike T, was not visible in it. It is interesting, as an example of how geometry moves on, that some sixty years later, the definition of Vecten's Point, and how it might be viewed changed, weakening the association with Euclid's Windmill, although, of course, the point remained the same (see Figure 12; and compare [17, p. 82], where the divorce from Figure 1 seems complete).

It seems to have been Charles Ange Laisant (1841–1920) who, in 1877, first explicitly redirected attention [10] to the centres A_1, B_1 and C_1 of the squares placed externally on the sides of $\triangle ABC$ opposite the respective vertices A, B and C (for some historical notes, see [3, 5th ed., esp. pp. 860–863], which also provides a general reference for geometrical results of the period). As in Section 6, introducing the centres of these squares presents us with cyclic quadrilaterals for deployment. Our arguments there show how bisectors of right angles are associated with the presence of isosceles right triangles in cyclic quadrilaterals. Applying this reasoning now, we see, for example, first of all that the circle on BC as diameter passes through A_0 and A_1, and then that A_1 lies on the angle bisector of the right angle $\angle BA_0C$. Hence, AA_0A_1, and similarly BB_0B_1 and CC_0C_1 are straight lines, so that the Vecten Point can be redefined as the common intersection of the lines AA_1, BB_1 and CC_1, without reference to construction lines considered by Euclid in proving *Elements I.47* (as in [17, p. 82]).

This alternative definition was quickly spotted by Joseph Jean Baptiste Neuberg (1840–1926), who realised further that it allowed the existence of Vecten's Point to be deduced from the concurrency of the altitudes of a triangle, since it is comparatively

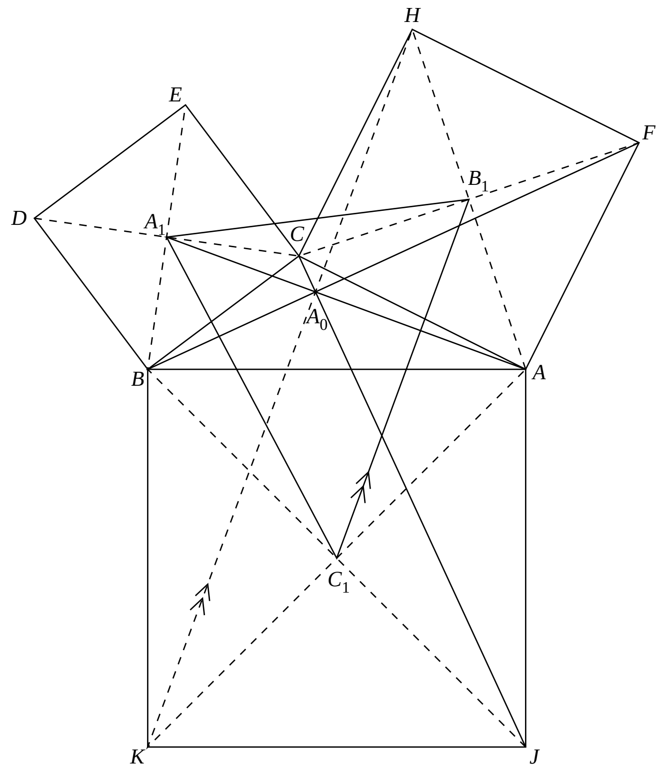


Figure 12: Vecten's point — construction

easy to identify AA_1, BB_1 and CC_1 as the altitudes of the triangle $\triangle A_1B_1C_1$ [13]. Despite the clear difference in emphasis, perhaps this is not so far from Vecten's own thinking. Vecten was aware, for example, that HA_0K is a straight line perpendicular to AA_0, with analogous results for EB_0J and DC_0F. Certainly, the reasoning in Section 6, applied to the cyclic quadrilaterals A_0CHF and A_0JKB, shows that H is on the angle bisector of the right angle $\angle CA_0F$, while K is on the angle bisector of the right angle $\angle BA_0J$, so confirming Vecten's observation — for what it is worth, H and K may be viewed, after the manner of Laisant, as the centres of square placed outwardly on CF and BJ. The fact that such pairs of lines are perpendicular then plays into Neuberg's redescription of the Vecten Point. Since B_1 and C_1 are the mid-points of AH and AK, the triangles $\triangle AHK$ and $\triangle AB_1C_1$ are in perspective from A (compare Figure 12). Thus, B_1C_1 being parallel to HK, we know from Vecten that it is perpendicular to AA_0, that is to AA_1, establishing Neuberg's altitude property. But then Vecten was modest, yet clear sighted, in self-appraisal [15, p. 322]:

> Je ne vous envoie pas les démonstrations de ces diverses propositions, parce qu'elles sont toutes extrément simples, et qu'elles se présentent, pour ansi dire, d'elles-mémes en constrisant la figure.

.

References

[1] J. Casey *A Sequel to the First Six Books of the Elements of Euclid* (Hodges, Figgis and Co., Dublin; Longmans, Green and Co., London, 1881).

[2] J. L. Coolidge, *A History of Geometrical Methods* (Clarendon Press, Oxford, UK, 1940; reprinted, Dover Pub., New York, NY, 1963).

[3] Fréres des Écoles Chrétienns, *Cours de Mathématiques Élémentaires: Exercises des Géometrie* (1st ed., 1875; 2nd ed., 1882; 3rd ed., 1896; 4th ed., 1907; 5th ed., Fréres Gabriel-Marie (Edmund Brunhes), ed., Mame, Tours and de Gigord, Paris, 1912; 6th ed., Éditions Jacques Gabay, Paris, 1991); 5th ed. available at ⟨`http://quod.lib.umich.edu/u/umhistmath`⟩.

[4] J. D. Gergonne, (a) Démonstration d'un théorème énoncé dans le Philosophical Magazine, pour septembre 1823, *Annales de mathématiques pures et appliquées (Gergonne's Annales)*, **14** (1823–1824), 334–336; (b) from B. D. C, Démonstration du théorème de M. Hamett, mentionné à la page 334 du présent volume, *ibid*, **14** (1823–1824), 374–375; (c) with P. J. J. Querret, Démonstration de deux théorèmes de géométrie, desquels on peut déduire, comme cas particulier, le théorème de M. Hamett, mentionné aux pages 334 et 374 du précédent volume, *ibid*, **15** (1824–1825), 84–89; all available at ⟨ `http://www.numdam.org`⟩.

[5] J. Hamett, *Philosophical Magazine*, (1823), 236.

[6] T. L. Heath, *The Thirteen Books of Euclid's Elements: Translated with Introduction and Commentary, 3 Vols.* (Cambridge University Press, Cambridge, UK, 1908, 2nd ed., 1926; reprinted, Dover Pub., New York, NY, 1956).

[7] T. L. Heath, *A History of Greek Mathematics, 2 Vols.* (Clarendon Press, Oxford, UK, 1921; reprinted, Dover Pub., New York, NY, 1981).

[8] L. Hoehn, Note 80.38: Pythagoras inside out, *Math. Gaz.*, **80** (1996), 544-545.

[9] C. Kimberling, (a) Triangle centers and central triangles, *Congressus Numerantium*, **129** (1998), 1–295; (b) *Encyclopedia of Triangle Centers*, available at ⟨ `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html` ⟩

[10] C.-A. Laisant, *Nouvelle Correspondance mathématique* (1877), 368; 400.

[11] A. Lo Bello, ed., *The commentary of Al-Nayrizi on Book I of Euclid's* Elements of geometry*. With an introduction on the transmission of Euclid's* Elements *in the Middle Ages* (Brill, Leiden, 2003).

[12] E.-E. Neël, *Théorèmes, lieux géometriques et problèmes: les applications de Blanchet* (Office de la publicité, Brussels, 1879); available at ⟨`http://gallica.bnf.fr`⟩.

[13] J. J. B. Neuberg, *Nouvelle Correspondance mathématique*, 4 (1878), 142–145.

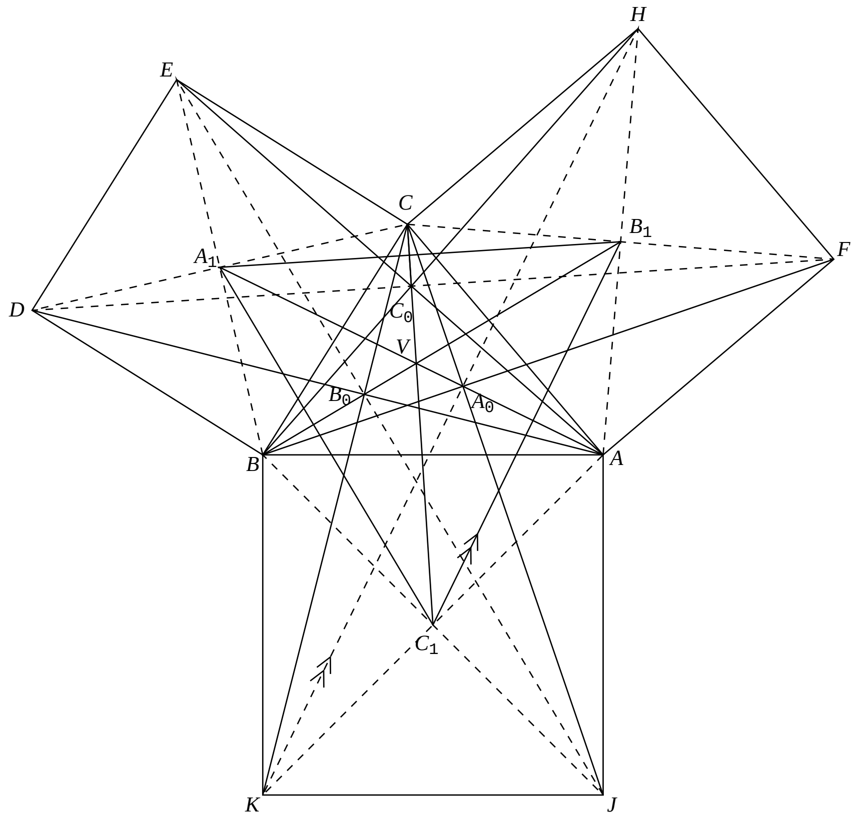


Figure 13: Vecten's point — in full

[14] M. R. F. Smyth, (a) The MacCool/West point, *Irish Math. Soc. Bull.*, **57** (2006), 93–97; (b) MacCool's proof of Napoleon's theorem: a sequel to the MacCool/West point, *ibid*, **59** (2007), 71–77.

[15] M. Vecten, Extrait d'une lettre au Rédacteur des Annales, *Annales de Mathématiques pures et appliquées (Gergonne's Annales)*, **7** (1816–1817), 321–324; available at ⟨`http://www.numdam.org`⟩.

[16] H. Vuibert, *Journal de mathématiques élémentaires* (1879–1880), 36.

[17] H. Walser, *99 Schnittpunkte: Beispiele – Bilder – Beweisse* (Edition am Gutenbergplatz, Leipzig, 2004), trans P. J. Hilton and J. J. Pedersen, as *99 Points of Intersection: Examples, Pictures, Proof* (Math. Assoc. Amer., Washington, DC, 2006).

[18] J. E. Wetzel, (a) Squares in triangles, *Math. Gaz.*, **86** (2002), 28–34; (b) Fits and covers, *Math. Mag.*, **76** (2003), 349–363.

[19] W. P. Workman, Geometry (paper read before Bath Branch, Teachers Guild, May, 1891), *The Practical Teacher*, (July, August, 1891); abridged as Chap. VIII in F. Spencer, ed., *Chapters in the Aims and Practice of Teaching* (Cambridge University Press, Cambridge, UK, 1897), pp. 193–207.