A Round-Up of Square Problems

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A Round-Up of Square Problems

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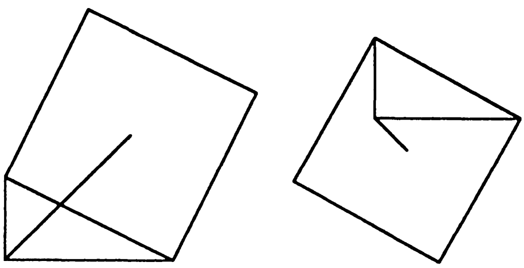
Introduction

Are squares, as their name suggests, really the boring "nerds" of the geometric world? We think not, and have gathered a number of our favorite problems that we hope show the square to be a fascinating figure. Many of the results deserve to be (and indeed are) theorems, but much of the fun seems to be in presenting the material in the form of problems. You are challenged to find your own solutions, and we hope you won't jump too quickly to the solutions we have provided.

The problems are, at least roughly, divided into sections according to the number of squares involved. Many of the problems are new, at least to us, and others may be familiar to some readers. Even then, some novelty is found in most of the solutions, and in several places we have uncovered unexpected connections among what at first may seem to be unrelated problems. A concluding section provides some sources, though it is not always easy to know who deserves first credit. There should be little harm in rediscovering a neglected gem, and much interest and pleasure to be gained.

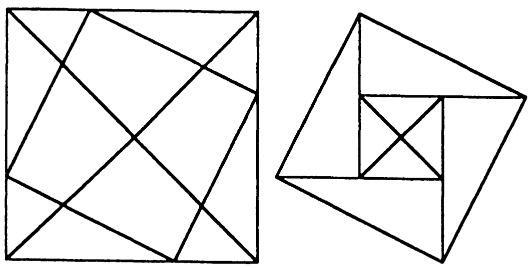
Problems About One Square

Problem 1. A square is erected, either externally or internally, on the hypotenuse of a right triangle. Show that the line segment from the vertex of the right angle to the center of the square makes 45° angles to the legs of the right triangle.

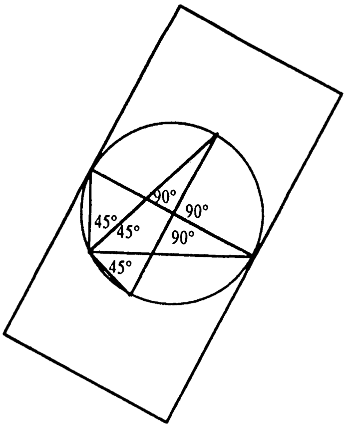


Solution 1. Here are two nice ways to solve this problem.

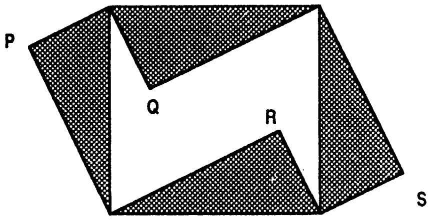
(**a**) *via tiling:* Adding congruent copies of the right triangle to the remaining sides of the given square gives us a second square that makes the result visually obvious. The segment through the center of the second square is along a diagonal of the new square, so it bisects the right triangle.



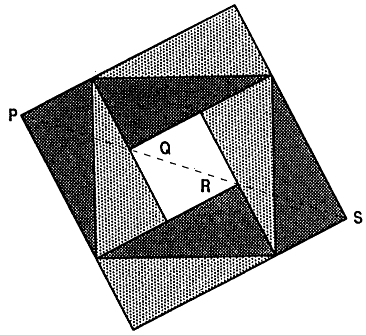
(b) *via the inscribed angle theorem:* Both the internally and externally erected square cases can be shown together. Construct the circle centered on the hypotenuse of the right triangle. The legs of the right triangle and the segments to the centers of the squares intercept 90° arcs on the circle. By the inscribed angle theorem, each inscribed angle has half the measure of the 90° arc, namely 45°.



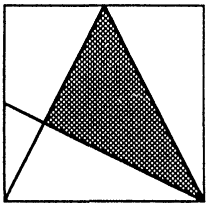
Problem 2. Congruent right triangles are erected to the sides of a square, facing alternately outward and inward as shown. Show that P, Q, R, and S are collinear.



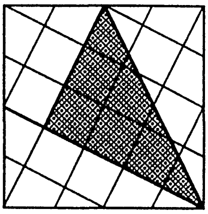
Solution 2. Combining the tilings shown in the solution of Problem 1 reveals that P, Q, R, and S are all on the diagonal of a circumscribed square.



Problem 3. The shaded triangle at the right is formed by drawing segments from corners of the square to the midpoints of opposite sides, as shown. Show that the triangle is a right triangle with sides in the proportion 3 : 4 : 5.

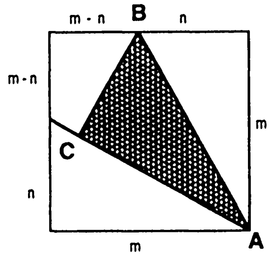


Solution 3. There is an elegant tiling solution, formed by overlapping the given figure in a square grid containing the points $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$ along the edges of the square.



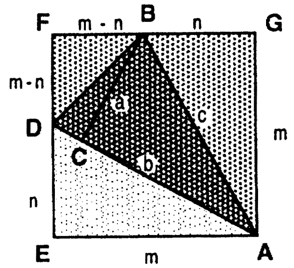
Problem 4. Let m and be positive integers with $m > n$. The shaded right triangle, ΔABC, is constructed in an m by m square as shown. Show that the triangle has sides in the integer proportions

$$(m^2 - n^2) \colon 2mn \colon (m^2 + n^2).$$



(*Note:* Choosing m and n relatively prime and of opposite parity, it is well known that all primitive Pythagorean triples are of the form $m^2 - n^2$, $2mn$, and $m^2 + n^2$. Thus the construction realizes all of the right triangles with integer sides in appropriately sized squares.)

Solution 4. The square $AEFG$ can be viewed as being dissected into four triangles, as shown. We then obtain the area equation



$$area(AEFG) = area(\Delta ADE) + area(\Delta ABG) + area(\Delta BFD) + area(\Delta ABD).$$

Letting

$$a = BC, b = AC, c = AB = AD = \sqrt{m^2 + n^2},$$

we see that the area equation becomes

$$m^2 = \frac{mn}{2} + \frac{mn}{2} + \frac{(m-n)^2}{2} + \frac{ac}{2}.$$

Solving for a we find $a = (m^2 - n^2)/c$, from which we learn that

$$b^2 = c^2 - a^2 = \frac{(c^4 - a^2c^2)}{c^2} = \frac{(c^2 - ac)(c^2 + ac)}{c^2}$$

$$= \frac{(m^2 + n^2 - m^2 + n^2)(m^2 + n^2 + m^2 - n^2)}{c^2} = \frac{4m^2n^2}{c^2}.$$

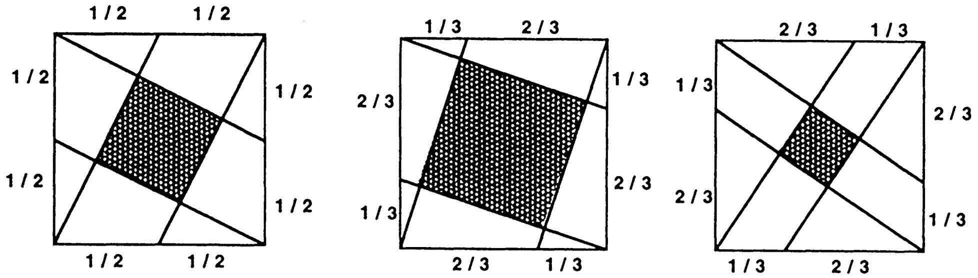
That is, $b = 2mn/c$. Writing $c = c^2/c = (m^2 + n^2)/c$ we see that

$$a = \frac{(m^2 - n^2)}{c}, b = \frac{2mn}{c}, c = \frac{(m^2 + n^2)}{c}.$$

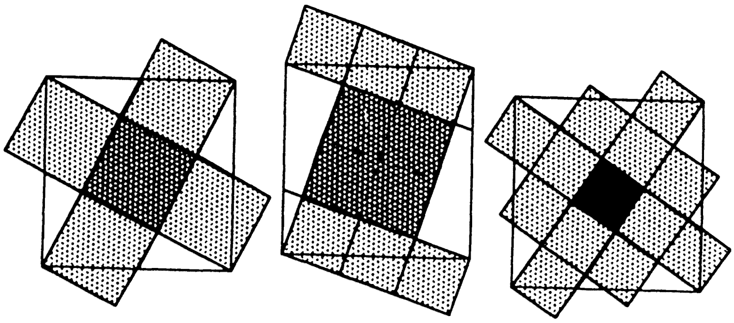
Therefore, $a : b : c = (m^2 - n^2) : 2mn : (m^2 + n^2)$.

Problems About Two Squares

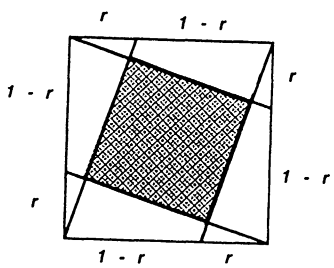
Problem 5. A square is created by connecting each vertex of the unit square to a point on a nonadjacent side, as shown in these three examples. What is the area of each shaded square?



Solution 5. In each case, an inscribed square grid makes the answers readily apparent as shown below. In each case the triangular regions lying in the exterior of the original unit square are paired with a congruent triangle within the unit square that lies outside the shaded square region. The dark-shaded squares are seen to have respective areas $\frac{1}{5}$, $\frac{4}{10}=\frac{2}{5}$, and $\frac{1}{13}$.

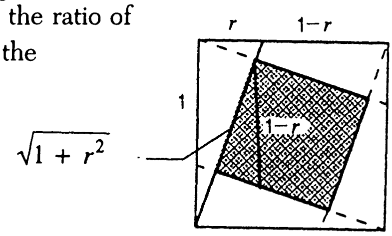


Problem 6. Find the area of the shaded square contained within the unit square as shown, where $0<r<1$.



Solution 6. A vertical segment drawn from a vertex of the shaded square to the opposite side has length $1-r$, compared to a length of $\sqrt{1+r^2}$ of a corresponding segment in the unit square. Thus the ratio of similarity is $(1-r)/\sqrt{1+r^2}$, making the area of the shaded square

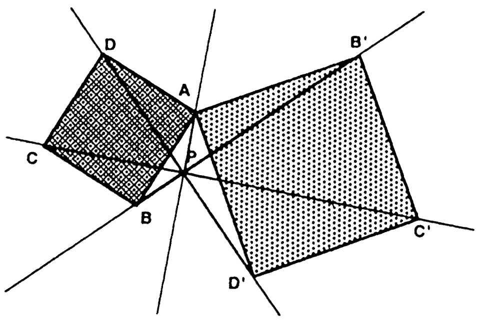
$$A=\frac{(1-r)^2}{1+r^2}.$$



For example, if $r=\frac{2}{3}$, then $A=\dfrac{\left(1-\frac{2}{3}\right)^2}{1+\left(\frac{2}{3}\right)^2}=\dfrac{\frac{1}{9}}{\frac{13}{9}}=\dfrac{1}{13}$

in agreement with the dissection solution shown in Problem 5.

Problem 7. Let the squares $ABCD$ and $AB'C'D'$ share a vertex at A, where both squares are labeled clockwise.

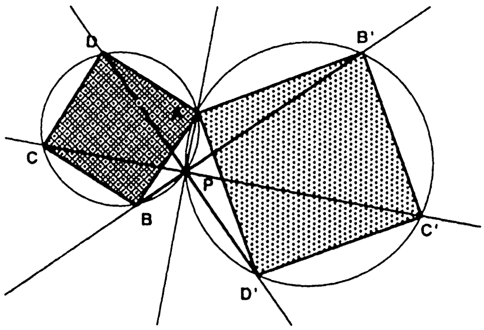


(a) Show that the segments BB' and DD' are the same length and lie on perpendicular lines.

(b) Let P be the point at which the perpendicular lines BB' and DD' intersect. Show that the line CC' also passes through P, and is an angle bisector.

(c) Show that the line AP is perpendicular to line CC'.

Solution 7. (a) A 90° rotation about point A transforms $\Delta ABB'$ onto $\Delta ADD'$, showing that the triangles are congruent. In particular, $BB'=DD'$ and are contained in lines that cross at 90°.

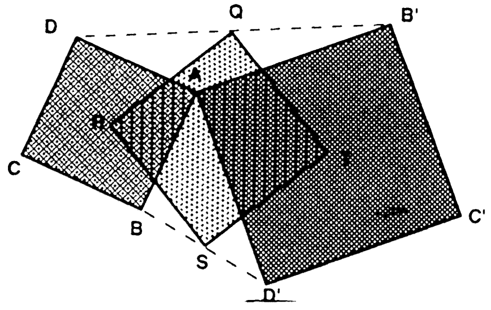


(b) Draw the circumscribing circles of each of the squares. These circles intersect at A and P so (cf. Solution 1(b)) by the inscribed angle theorem we see that PC and PC' are each angle bisectors of the right angles at P.

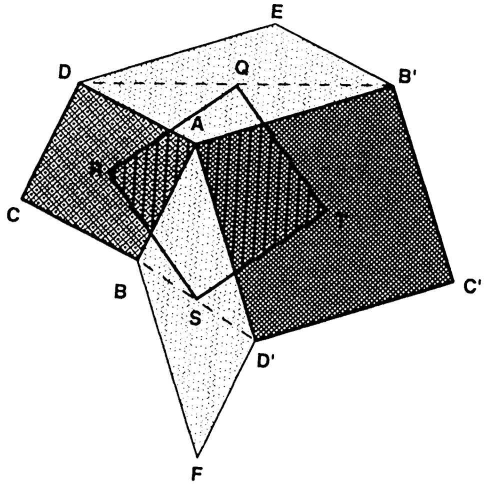
(c) Since the rays PA and PC intercept diametrically opposite points A and C of the circumscribing circle, $\angle APC$ is a right angle.

Remark. An alternate proof of parts (b) and (c) can be based on the results of part (a) and Problem 1.

Problem 8. Let squares $ABCD$ and $AB'C'D'$ share a vertex (as in Problem 7). Show that the midpoints, Q and S, of the segments $B'D$ and BD' together with the centers R and T of the squares form another square, $QRST$.

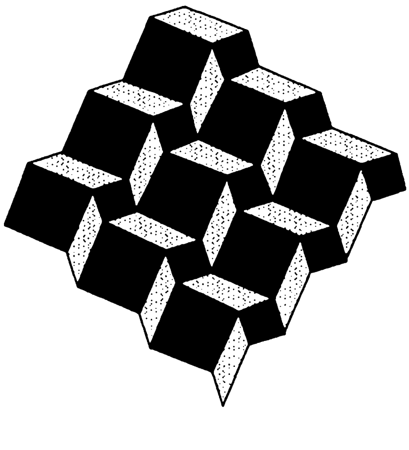


Solution 8. A pair of congruent parallelograms, $AB'ED$ and $BFD'A$ have Q and S as their respective centers. Since a 90° rotation about R transforms $AB'ED$ onto $BFD'A$ we see that RS and RQ are congruent segments meeting at 90°. Similarly, QT and ST are congruent and orthogonal, so it follows easily that $QRST$ is a square.



Remark 1: This result is sometimes known as the *Finsler-Hadwiger* theorem. It will be convenient later to refer to $QRST$ as the *Finsler-Hadwiger square* determined by the given squares sharing a vertex at point A. Note that the entire configuration is uniquely determined by the three points A, R, and T.

Remark 2: A visualization of the generation of the Finsler-Hadwiger squares is provided by tiling the plane with the octagon shown above. The centers of the parallelograms and squares are seen to form a square grid. A second square grid, of twice the linear size, is formed by the translates of the square $CEC'F$. This same tiling can also be used to visualize the results of Problem 7.



Remark 3: The result of Problem 8 is actually a special case of a more general theorem that is elementary yet of interest.

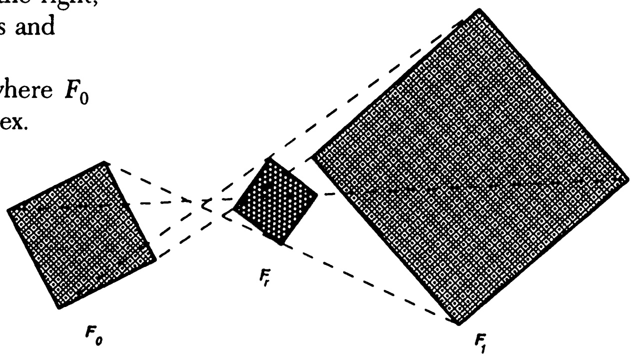
THEOREM. *Let F_0 and F_1 denote two directly similar figures in the plane, where $P_1 \in F_1$ corresponds to $P_0 \in F_0$ under the given similarity. Let $r \in (0,1)$, and define $F_r = \{(1-r)P_0 + rP_1 \colon P_0 \in F_0\}$. Then F_r is also directly similar to F_0.*

Proof. We assume the figures are in the complex plane, so that the similarity has the form $z \mapsto az + b$, where a and b are complex constants with $a \neq 0$. Thus F_0 is mapped to F_r by the map

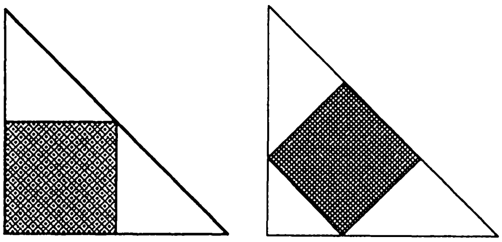
$$\sigma_r(z) = (1-r)z + r(az+b) = (1-r+ra)z + rb,$$

which has the form of a direct similarity transformation. ∎

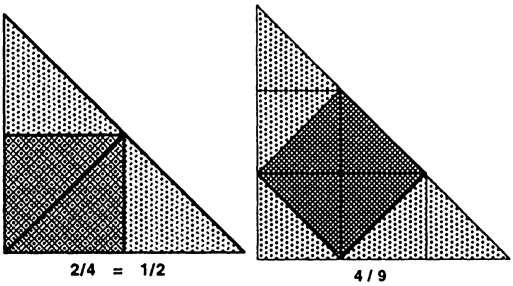
An example of the theorem is illustrated in the diagram at the right, where the figures are squares and $r = \frac{1}{2}$. The Finsler-Hadwiger theorem is the special case where F_0 and F_1 share a common vertex.



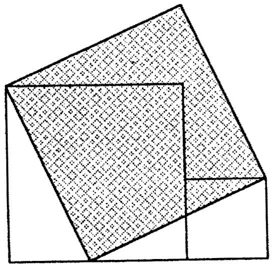
Problem 9. Squares have been inscribed in congruent isosceles right triangles in two different ways. Which square has the larger area?



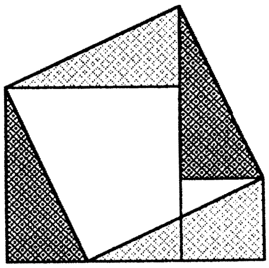
Solution 9. Triangular grids show that the respective areas are $\frac{2}{4}$ and $\frac{4}{9}$. Thus $\frac{1}{18}$, or about $5\frac{1}{2}\%$, more of the triangle's area is covered by the square on the left.



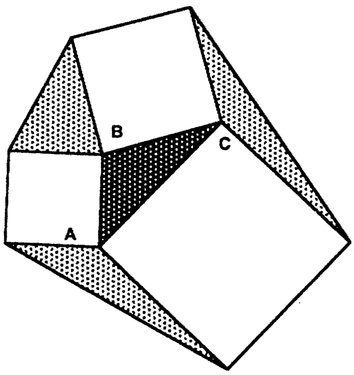
Problems About Three Squares



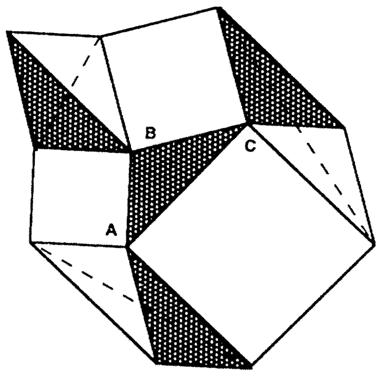
Problem 10. Two side-by-side squares are constructed on a horizontal segment. The upper left-most and right-most vertices are then used as opposite vertices of a tilted larger square. Show that the large square has one vertex on the horizontal segment and another vertex on the extension of the common vertical sides of the small squares. Then compare the areas of the three squares.



Solution 10. The configuration described in the problem statement is a thinly-disguised confirmation of the Pythagorean theorem, which is surely the most famous result about three squares in all of geometry. The dissected figure at the right makes it visually clear that the area of the large square is the sum of areas of the two smaller squares. The dissection is attributed to Tâbit ibn Qorra (826–901), and was rediscovered in 1873 by Henry Perigal.

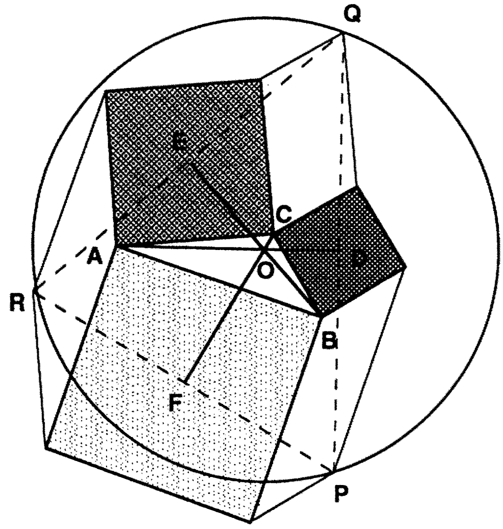


Problem 11. Given any triangle ABC, erect outward facing squares on all three sides. Three additional triangles are then constructed, as shown in the figure. Show that all four triangles have the same area.

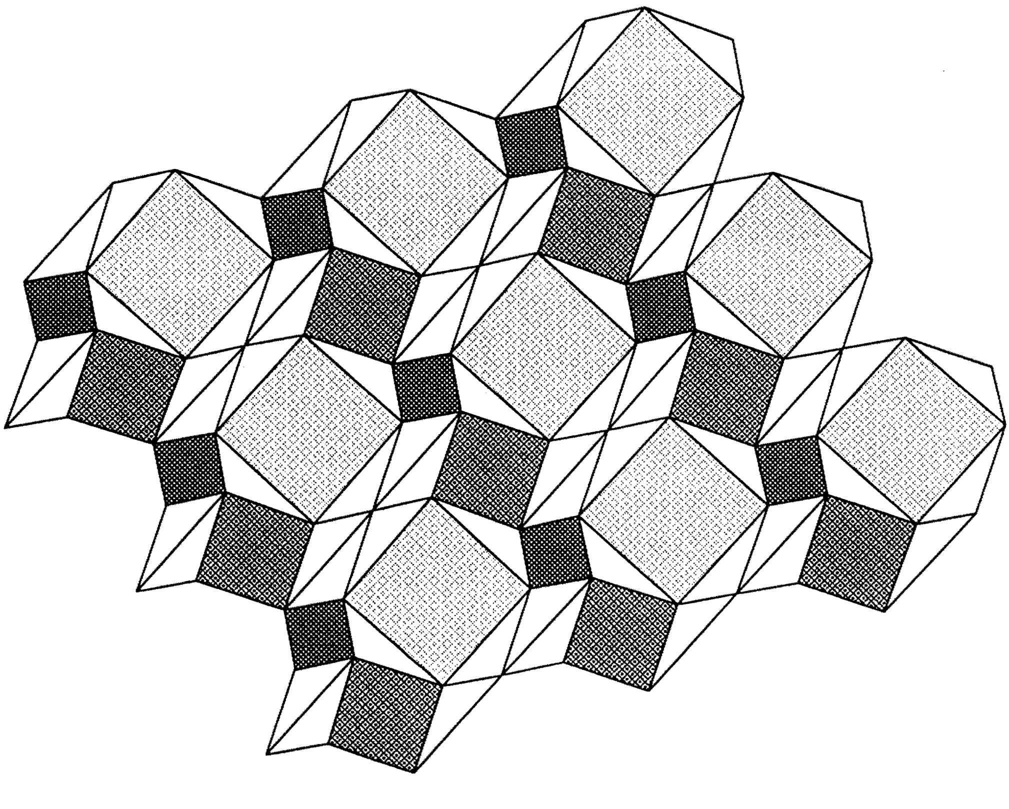


Solution 11. The tiling shown in the solution to Problem 8 provides a simple way to see why the triangles have equal area: extend each outer triangle to a parallelogram. Drawing the opposite diagonal forms triangles that are all congruent to ΔABC, and therefore have areas equal to the original triangle of the problem.

Problem 12. Outward facing squares with centers D, E, and F are erected on the sides of an arbitrary triangle ABC. Next, parallelograms are constructed as shown, determining P, Q and R. Show that the segments AD, BE, and CF are concurrent at a point O that is the center of the circumscribed circle of ΔPQR.



Solution 12. The result is evident in the beautiful tiling shown below. For example, we easily see that 90° and $-90°$ rotations about D will take the point A to P and Q, respectively. Thus AD is the perpendicular bisector of PQ.



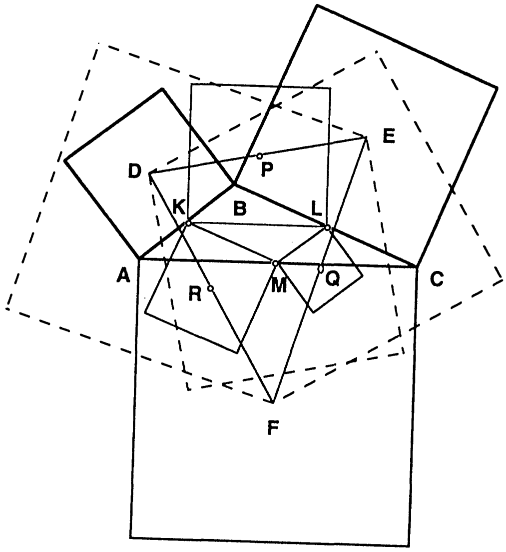
Problems About Four or More Squares

Problem 13. Let squares be erected externally on the sides of a triangle ABC, with centers D, E, and F.

(a) Show that the midpoints K, L, and M of the sides of ΔABC coincide with the centers of the squares erected internally on the sides of triangle DEF.

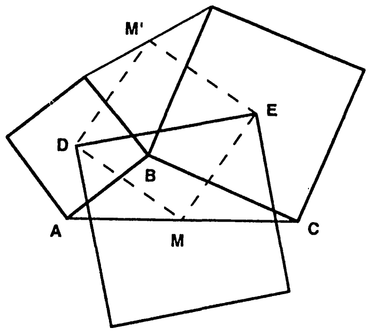
(b) Show that the centers P, Q, and R of the squares erected externally on the sides of ΔKLM coincide with the midpoints of the sides of ΔDEF.

The properties also hold if internally and externally are interchanged.

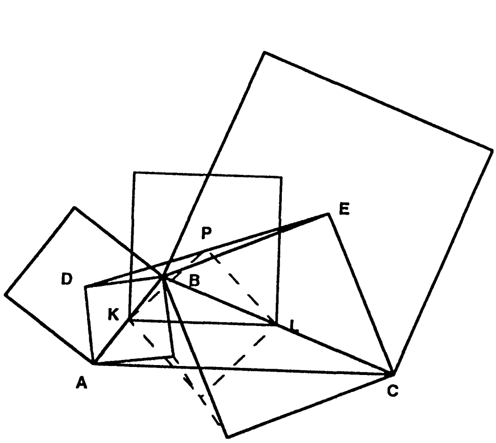


Solution 13.

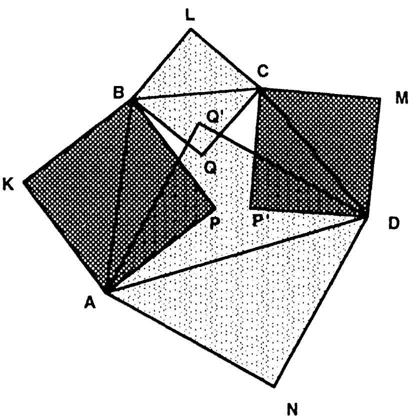
(a) The Finsler-Hadwiger square (shown dashed) determined by the squares centered at D and E (see *Remark 1* in Solution 8) has one vertex at M, the midpoint of side AC. But M is also seen to be the center of the square with side DE.

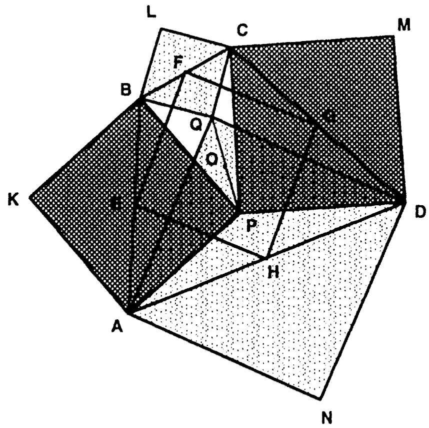


(b) Construct squares with diagonals AB and BC. The Finsler-Hadwiger square corresponding to the squares whose diagonals are AB and BC has P, the midpoint of DE, as a vertex. Clearly P is also the center of the externally erected square on side KL.



Problem 14. Construct squares whose diagonals are the sides of a quadrilateral $ABCD$. Let K, L, M, and N denote the external vertices, and P, Q, P', and Q' the internal vertices, of the squares, as shown at the left below.



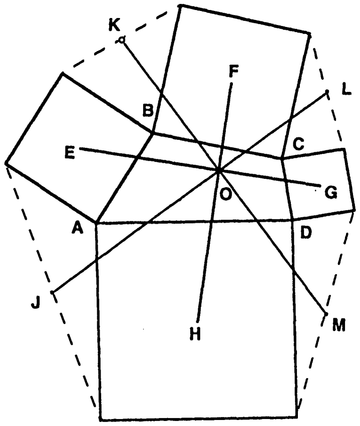


(a) Show that $P = P'$ if and only if $Q = Q'$, as shown above on the right.
(b) In the case that $P = P'$ (and $Q = Q'$), show that:

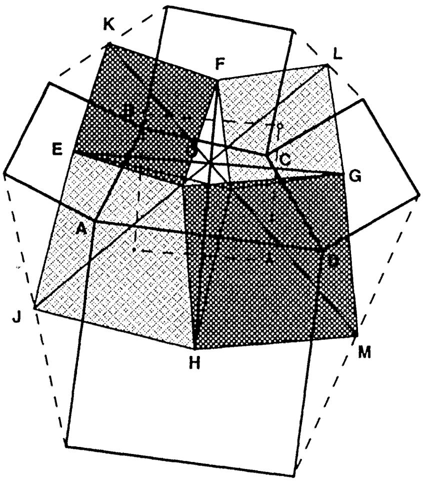
- the midpoints of the sides of $ABCD$ form a square, $EFGH$;
- the center, O, of square $EFGH$ is also the midpoint of segment PQ;
- the sum of the areas of the two squares sharing vertex P is equal to the sum of the areas of the two squares sharing vertex Q.

Solution 14. Suppose that $P = P'$. Then the squares with diagonals AB and CD generate the Finsler-Hadwiger square $EFGH$, which has its vertices at the midpoints of the sides of $ABCD$. By Remark 1 following Solution 8, there is a unique square centered at F which, together with the square $AQ'DN$, generates $EFGH$ as their corresponding Finsler-Hadwiger square. But this square, centered at F, has diagonal BC, so $Q = Q'$. By the result in Problem 13(a) (which is *Neuberg's theorem*), applied to ΔBQP, we deduce that the center O of the square with side EF is at the midpoint of QP. By the result of Problem 7, the diagonals AC and BD lie on perpendicular lines; thus, as easily follows from the Pythagorean theorem, $AB^2 + CD^2 = BC^2 + DA^2$. This equation shows that the sums of the areas of opposite squares are equal.

Problem 15. Squares are erected externally on the sides of quadrilateral $ABCD$, with centers E, F, G and H. Show that the segments EG and FH are congruent and lie on perpendicular lines. Similarly, if J, K, L, and M are the midpoints of the dashed segments shown, prove that JL and KM are congruent segments that lie on perpendicular lines, with the length of these segments $\sqrt{2}$ times the length of EG and FH. Moreover, show that all four lines are concurrent, intersecting at point O at 45° angles.



Solution 15. The configurations discovered in some of the preceding problems provide the keys. By Problem 13, the squares with diagonals EF and GH have a common vertex at the midpoint of AC, as we see in the figure at the right. Similarly, the squares with diagonals FG and EH have a common vertex at the midpoint of BD. Problem 7 showed us that EG and FH are congruent and lie on perpendicular lines, that the same property holds for JL and KM, and that all four lines are concurrent. Moreover, the common length of EF and GH is twice the length of the side of the Finsler-Hadwiger square (shown dashed) formed by the centers of the newly constructed squares. Similarly, the common length of JL and KM is twice the length of the diagonal of the Finsler-Hadwiger square.



Sources and Additional Remarks for Selected Problems

Problem 1 was inspired by a problem of Larry Hoehn **[8]**. The case of the internally erected square, and the inscribed angle proof, are apparently new. The first case of Problem 5 is attributed to Heinrich Dörrie by Edward Kitchen **[9]**. Kitchen's article solves the second case with a different tiling than ours, and also discusses a number of similar problems dealing with squares. Problem 7 (b) and (c), in a slightly different form, appeared as the first two parts of a problem of Andrew Cusumano **[2]**; his references indicate that the problem has reappeared several times beginning in 1919. A solution to the problem in **[11]** is similar to ours.

The result of Problem 8, which introduced the Finsler-Hadwiger square **[4]**, is proved in **[5]** in a very different way; **[5]** also contains a list of references, supplied by Murray Klamkin, related to the Finsler-Hadwiger theorem. The tiling shown in Remark 2 seems to be a new connection to the theorem (in a strange coincidence, almost to the day the tiling was first drawn, the same pattern was seen worn on a tie by comedian Tim Allen in the popular television show *Home Improvement!*). The result of Remark 3 was a rediscovery of what Howard Eves calles the *fundamental theorem of directly similar figures* **[3]**; the application to the Finsler-Hadwiger theorem seems to be new.

Problem 9 was contributed by James Varnadore **[12]** as a calendar problem, but the simple dissection proof we have given is new. Problem 11 is due to Bishnu Naraine **[10]**, who gives a trigonometric solution. A letter of Bo Burbank **[1]** gives the beautiful transformational proof we have reproduced. Problem 12 and the tiling shown in the solution seem to be new. Part (a) of Problem 13 is due to Joseph Neuberg (1840–1926); see **[7]**. Problem 14 (a) is a variant of the Douglas-Neumann theorem, discovered independently by Jesse Douglas and B. H. Neumann in 1940; see **[3]** for references. The first part of Problem 15, which shows the congruence and orthogonality of the segments connecting opposite erected squares on a quadrilateral, is the well-known theorem of von Aubel (see **[9]**, for example, for a vector proof). The extensions in Problems 13, 14, and 15 seem to be new, as are the connections to the Finsler-Hadwiger theorem in those problems.

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