From Lehrer, R. & Chazan, D. (Eds). (1998). Designing Learning Environments for developing Understanding of Geometry and Space. Mahwah, NJ: Lawrence Erlbaum Publishers, pp. 369-393.

15

An Alternative Approach to Proof in Dynamic Geometry

Michael de Villiers University of Durban-Westville

[There is an underlying] formalist dogma that the only function of proof is that of verification and that there can be no conviction without deductive proof. If this philosophical dogma goes unchallenged, the critic of the traditional approach to the teaching of proof in school geometry appears to be advocating a compromise in quality: he is a sort of pedagogic opportunist, who wants to offer the student less than the 'real thing.' The issue then, is not, what is the best way to teach proof, but what are the different roles and functions of proof in mathematics. (adapted from Hersh, 1979, p. 33)

The problems that students have with perceiving a need for proof are well known to all high-school teachers and have been identified without exception in all educational research as a major problem in the teaching of proof. Who has not yet experienced frustration when confronted by students asking "Why do we have to prove this?" Gonobolin (1954/1975) noted that "the students ... do not ... recognize the necessity of the logical proof of geometric theorems, especially when these proofs are of a visually obvious character or can easily be established empirically" (p. 61).

The recent development of powerful new technologies such as Cabri-Geometre and Geometer's Sketchpad with drag-mode capability has made possible the continuous variation of geometric configurations and allows students to quickly and easily investigate the truth of particular conjectures. What implications does the development of this new kind of software have for the teaching of proof? How can we still make proof meaningful to students?

In this chapter, a brief outline of the traditional approach to the teaching of proof in geometry is critiqued from a philosophical as well as a psychological point of view, and in its place an alternative approach to the teaching of proof (in a dynamic geometry environment) is proposed.

THE TRADITIONAL APPROACH TO THE TEACHING OF PROOF IN GEOMETRY

Underlying Philosophy

Philosophically, the traditional view of proof has been and still is largely determined by a kind of philosophical rationalism, namely, that the formalist view that mathematics in general (and proof in particular) is absolutely precise, rigorous, and certain. Mathematics from this perspective is seen as the science of rigorous proof. Although this rationalistic view has been strongly challenged in recent years by the quasi-empirical (or fallibilist) views of, for example, Lakatos (1976), Davis and Hersh (1983, 1986), Chazan (1990), and Ernest (1990), it is probably still held by the vast majority of mathematics teachers and mathematicians.

In an extreme version of this view, the *function* (or purpose) of proof is seen as only that of the verification (conviction or justification) of the correctness of mathematical statements. In other words, proof is narrowly seen merely as a means to remove personal doubt and/or that of skeptics, an idea that has one-sidedly dominated teaching practice and most discussions and research on the teaching of proof (even by those who profess to oppose a formalist philosophy). Consider the following quotes, which only emphasize the verification function of proof:

- "A proof is only meaningful when it answers the student's *doubts*, when it proves what is not obvious" (emphasis added; Kline, 1973, p. 151).
- "The necessity, the functionality, of proof can only surface in situations in which the students meet *uncertainty* about the truth of mathematical propositions" (emphasis added; Alibert, 1988, p. 31).
- "A proof is an argument needed to *validate* a statement, an argument that may assume several different forms as long as it is *convinc-ing*" (emphasis added; Hanna, 1989, p. 20).
- "Why do we bother to prove theorems? I make the claim here that the answer is: so that we may *convince people* (including ourselves). ... we may regard a *proof as an argument sufficient to convince a reasonable skeptic*" (emphasis added; Volmink, 1990, pp. 8, 10).

Arguing from the viewpoint that the results of all inductive or quasiempirical investigation are unsafe, proof is seen (basically) as a prerequisite for conviction—therefore, proof is required as the absolute guarantee of their truth. In other words, the only purpose of proof is to give the final stamp of approval:

Reasoning by induction and analogy calls for recourse to observation and even experiment to obtain the facts on which to base each argument. But the senses are limited and inaccurate. Moreover, even if the facts gathered for the purposes of induction and analogy are sound, these methods do not yield unquestionable conclusions. ... To avoid these sources of error, the mathematician utilizes another method of reasoning. ... In deductive reasoning the conclusion is a logically inescapable consequence of the known facts. (Kline, 1984, pp. 11– 12)

Traditional Teaching Approach

Because proof is seen only as an instrument for the removal of doubt, the typical approach is to create doubts in the minds of students, thereby attempting to motivate a need for proof. Traditional approaches in motivating proof in geometry can be classified into three main types, namely, pattern failure, optical illusion, and false conclusion.

Type 1: Pattern Failure. Pattern failure is often the "easiest" (most visual) way to create doubts:

- $n^2 n + 41$ gives prime numbers for n = 1, 2, 3, ... but breaks down when n = 41.
- *n* points on a circle, when all are connected, divide it into 2^{n-1} regions. This holds for n = 1 to n = 5, but breaks down when n = 6 (see Fig. 15.1).

There are also famous historical examples where inductive generalizations eventually turned out false. About 500 BC, Chinese mathematicians (and much later also Leibniz) conjectured that if $2^n - 2$ is divisible by n, then n must be prime. It turns out that the empirical investigation supports the conjecture up to $2^{340} - 2$. In all these cases $2^n - 2$ is divisible by nwhen n is prime, and not divisible by n when it is composite. However, in 1819 it was discovered that $2^{341} - 2$ is divisible by 341, even though 341 is composite ($341 = 11 \times 31$).

In 1984 Odlyzko and Te Riele showed that a conjecture by Franz Mertens (a contemporary of Riemann) over 100 years ago was actually false, despite computer support that showed that it was true up to $n = 10^7$.

After giving some such examples to students, teachers are usually satisfied that a sufficiently critical attitude has been cultivated and proceed to introduce the proofs of geometrical results, as a means of verifying that those results are correct.

Type 2: Optical Illusion. Another traditional approach is to provide students with optical illusions in order to caution them against putting too



FIG. 15.1 Pattern failure approach of proof. Given *n* points on a circle, all connected, $2^{n-1} =$ number of regions created. Pattern holds if $1 \le n \le 5$ but breaks at n = 6.

much faith in the way a figure looks. The intention is to show the "superiority of reasoning over experience." Consider the two examples in Fig. 15.2a. In both figures, although AB appears to be shorter than CD, AB and CD are, in fact, of equal length. Another often used example (see Fig. 15.2b) is an 8×8 square divided into four pieces, cut out and rearranged to form a 5×13 "rectangle" whose area is now suddenly 1 unit "greater" than that of the original square. What happens here is that actually a small narrow parallelogram with area of 1 unit is formed in the middle of the figure on the right, and the four pieces only appear to form a rectangle.

After some such examples, students are assumed to be convinced about the dangers of visual observation, and proof is then introduced as the "safe and sure" means of validating geometric statements that students have already confirmed experimentally.

Type 3: False Conclusion. Another approach is to give students a diagram like that in Fig. 15.3a for any arbitrary triangle ABC, followed by a "proof" that it then follows that CA = CB (i.e., that any triangle is isosceles). Children are then told that the obviously false conclusion arises because of the inaccuracy of the diagram (one of the points D and F always falls inside and the other outside, as shown in Fig. 15.3b), and are cautioned to be careful about how a figure looks or is drawn—that, in fact, all sketches are essentially unreliable and we should only rely on our power of reasoning.



FIG. 15.2 Optical illusion approach to proof. (a) AB = CD. (b) Greater area.



FIG. 15.3 False conclusion approach to proof.

A CRITICISM OF THE TRADITIONAL APPROACH TO THE TEACHING OF PROOF IN GEOMETRY

In what follows the author first criticizes the three types of examples given in the traditional teaching approach just given, before giving a more general critique of the underlying philosophy.

A Criticism of the Traditional Teaching Approach

The problem with the Type 1 examples given is that they are all actually from number theory, and not geometry at all (note that the second example is merely disguised as geometry). Although such examples are very appropriate in motivating proof within the context of number theory, their potential for motivating proof specifically within geometry is highly questionable. The author has yet to see a geometric configuration that has an invariant property for a very large number of cases (e.g., $n = 10^{10}$), but then suddenly breaks down for n + 1 cases!

It is furthermore important to note that there are subtle qualitative differences between such number-theoretic examples and the results of elementary geometry. First, the variables involved in the former are discrete, but the variables are continuous in the latter (e.g., angles, lengths). This is particularly the strong feature of drag-mode software like Cabri (Laborde & Bellemain, 1994) and Sketchpad (Jackiw, 1994), namely, that they allow for the *continuous* arbitrary variation and evaluation of geometric configurations. (Strictly speaking, these drag-mode transformations are only extremely good simulations of continuous variation, because the computer can calculate only discretely. More correctly, these variations are *near-continuous*.)

Second, many of the elementary geometry theorems are far more self-evident than these number-theoretic examples and can even be explained informally. Although students may not be able to articulate these subtle differences, many do sense it and are not necessarily convinced by such number-theoretic examples of the need for proof (as a means of verification) in geometry. (Many children of course quickly learn to play the teacher's game and start subscribing to this view only to please the teacher. After many years of "indoctrination" of this type, it sometimes takes a lot of deep probing to uncover children's actual personal views in this regard.)

The Type 2 examples (optical illusions) are deceitful: They do not encourage a need for deductive proof at all. For example, in the first two cases children are told to actually *measure* the lengths of AB and CD, finding them in fact to be equal. These examples therefore encourage measurement as the appropriate means of conviction/verification.

One of the famous French mathematicians once said, "Geometry is the art of drawing correct conclusions from incorrectly drawn sketches." But the false conclusion in the Type 3 example shows how easily a (correct) logical argument can lead to a fallacy because of a construction error or a mistaken assumption in a sketch. Instead of motivating a need for proof, such examples actually emphasize the importance of quasiempirical testing (i.e., the *accurate* construction of some examples).

These strategies of attempting to raise doubts in order to create a need for proof are even less likely to be successful when geometric conjectures have been thoroughly investigated through their continuous variation with drag-mode software like Cabri or Sketchpad (see Olive, chap. 16, this volume). When students are able to produce numerous corresponding configurations easily and rapidly then they simply have no (or very little) need for further conviction/verification. The problem is further intensified by a facility on Cabri that enables students to check whether certain features of configurations such as concurrency, collinearity, parallelness, perpendicularity, and equality of lengths are true in general. If not true in general, this facility produces a counterexample shown on screen. The computer, functioning as a "proof machine," reduces (in effect, eliminates) the students' need for self-generated proof (verification).

A Criticism of the Underlying Philosophy

As pointed out by Bell (1976), the traditional view of verification/ conviction being the main (or only) function of proof "avoids consideration of the real nature of proof [because conviction in mathematics is often obtained] by quite other means than that of following a logical proof" (p. 24). Research mathematicians, for instance, do not often scrutinize published proofs in detail, but are rather led by the established authority of the author, the testing of special cases, and an informal evaluation whether "the methods and result fit in, seem reasonable" (Davis & Hersh, 1986, p. 67).

With very few exceptions, teachers of mathematics seem to believe that a proof, for the mathematician, provides absolute certainty and that it is therefore the absolute authority in the establishment of the validity of a conjecture. They seem to hold the naive view described by Davis and Hersh (1986) that behind each theorem in the mathematical literature there stands a sequence of logical transformations moving from hypothesis to conclusion, absolutely comprehensible, and irrefutably guaranteeing truth. This view, however, is false. Proof is not necessarily a prerequisite for conviction—to the contrary, conviction is probably far more frequently a prerequisite for the finding of a proof.

A mathematician simply does not think: "Hmm ... this result looks very doubtful and suspicious; therefore, let's try to prove it." For what other weird or obscure reasons, would we then sometimes spend months or years to prove certain conjectures, if we weren't already reasonably convinced of their truth?

Polya (1954) wrote, for example, that

having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial *suspicion* and gave us a strong *confidence* in the theorem. Without such *confidence* we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is *true*, you start *proving* it. (emphasis added, pp. 83–84)

In situations where conviction provides the motivation for looking for a proof, the function of an eventual proof for the mathematician clearly cannot be that of verification/conviction, but has to be looked for in terms of explanation, discovery, communication, systematization, selfrealization, and so forth.

Absolute certainty also does not exist in real mathematical research, and personal conviction usually depends on a combination of intuition, quasi-empirical verification, and the existence of a logical (but not necessarily rigorous) proof. In fact, a very high level of conviction may sometimes be reached even in the absence of a proof. For instance, in their discussion of the "heuristic evidence" in support of the still unproved twin prime pair theorem and the famous Riemann hypothesis, Davis and Hersh (1983) concluded that this evidence is "so strong that it carries conviction even without rigorous proof" (p. 369).

That conviction for mathematicians is not reached by proof alone is also strikingly borne out by the remark of a previous editor of the Mathematical Reviews that approximately one half of the proofs published in it were incomplete and/or contained errors, although the theorems they were purported to prove were essentially true (Hanna, 1983, p. 71). It therefore seems that the reasonableness of results often enjoys priority over the existence of a completely rigorous proof. It is furthermore a commonly held view among today's mathematicians that there is no such thing as a rigorously complete proof (see Hanna, 1983, 1989; Kline, 1982). First, there is the problem that no absolute standards exist for the evaluation of the logical correctness of a proof nor for its acceptance by the mathematical community as a whole. Second, as Davis and Hersh (1986) pointed out, mathematicians usually only publish those parts of their arguments that they deem important for the sake of conviction, thus leaving out all routine calculations and manipulations, which can be done by the reader. Therefore a "complete proof simply means proof in sufficient detail to convince the intended audience" (Davis & Hersh, 1986, p. 73).

In addition, attempts to construct rigorously complete proofs lead to such long, complicated proofs that an evaluative overview becomes impossible and at the same time the probability of errors becomes dangerously high. For example, Manin (1981, p. 105) estimated that rigorous proofs of the two Burnside conjectures would run to about 500 pages each, and a complete proof for Ramanujan's conjecture would run to about 2000 pages. Even the proof for the well-known, but relatively simple, theorem of Pythagoras would take up at least 80 pages, according to Renz (1981, p. 85).

Limitative theorems by Gödel, Tarski, and others during the early part of this century have highlighted the inadequacy of the axiomatic method in general (and deductive proof in particular) for establishing firm foundations for the whole of mathematics. Lakatos (1976, 1978) also argued, from an epistemological analysis of examples from the history of mathematics, that proof can be *fallible*, and that it does not necessarily provide an absolute guarantee for the attainment of certainty:

There have been considerable and partly successful efforts to simplify Russell's *Principia* and similar logistic systems. But while the results were mathematically interesting and important they could not retrieve the lost philosophical position. The *grandes logiques* cannot be proved true—nor even consistent; they can only be proved false—or even inconsistent. (Lakatos, 1978, p. 31)

When investigating the validity of an unknown conjecture, mathematicians normally not only look for proofs, but also try to construct counterexamples at the same time by means of quasi-empirical testing because such testing may expose hidden contradictions, errors, or unstated assumptions. Frequently, the discovery/construction of counterexamples necessitates a reconsideration of old proofs and the construction of new ones. Personal certainty consequently also depends on the continued absence of counterexamples in the face of quasi-empirical evaluation. More generally, the attainment of personal conviction depends on positive justification and/or negative falsification (see Fig. 15.4).



FIG. 15.4 Underpinnings of personal conviction.

THE COMPUTER AS A MEANS OF EXPLORATION AND VERIFICATION

Recent years have seen an explosion in the use of computers as a means of exploration and verification in many areas of mathematics:

We find ourselves examining on the machine a collection of special cases which is too large for humans to handle by conventional means. The computer is encouraging us to practice unashamedly and in broad daylight, certain customs in which we indulge only in the privacy of our offices, and which we never admitted to students: experimentation. To a degree which never appears in the courses we teach, mathematics is an experimental science.... The computer has become the main vehicle for the experimental side of mathematics. (Pollak, 1984, p. 12)

But typically, the question arises: "How do we know that the computer has not made a mistake?" As Appel and Haken (1984), however, pointed out:

When proofs are long and highly computational, it may be argued that even when hand checking is possible, the probability of human error is considerably higher than that of machine error; moreover, if the computations are sufficiently routine, the validity of programs themselves is easier to verify than the correctness of hand computations. (p. 172)

Grünbaum (1993), in talking about his computer proof of a result from Euclidean geometry, presented an interesting argument based on work by Davis (1977) that the probability of his (Grünbaum's) findings being false are, for all practical purposes, zero:

The question arises what is the character of the facts I have been discussing? Do we start trusting numerical evidence (or other evidence produced by computers) as proofs of mathematics theorems? ... Is there any consequence to be drawn from the fact that in example after example numerical evidence establishes the homothety of Q and Q2? ... If we have no doubt—do we call it a theorem? ... I do think that my assertions about quadrangles and pentagons are theorems.... the mathematical community needs to come to grips with the possibilities of new modes of investigation that have been opened up by computers. (p. 8)

This, of course, raises a serious question: With the increasing use of the computer as a means of verification, is there still a need for a deductive proof? Of course, if we see the function of a deductive argument as only that of verification, we might as well now start saying, as Horgan (1993) did, that "proof is dead or dying" and bury it. However, as discussed in the next section, a deductive proof is useful for other reasons.

PROOF AS A MEANS OF EXPLANATION AND DISCOVERY

Although it is possible to achieve confidence in the general validity of a conjecture through the use of dynamic geometry features like property checkers and the drag-mode effect of continuous transformation across the screen, such features offer no satisfactory explanation why a given conjecture may be true. The software merely confirms that the conjecture is true, and even though the consideration of more and more examples may fortify a student's confidence, it gives no psychologically satisfactory sense of illumination—no insight or understanding into how a given conjecture is the consequence of other familiar results. Despite the convincing heuristic evidence in support of the earlier mentioned Riemann hypothesis, for example, we may have a need for explanation:

It is interesting to ask, in a context such as this, why we still feel the need for a proof.... It seems clear that we want a proof because ... if something is true and we can't deduce it in this way, this is a sign of a lack of understanding on our part. We believe, in other words, that a proof would be a way of understanding *why* the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reasoning that it is true. (Davis & Hersh, 1983, p. 368)

Gale (1990) also clearly emphasized, with reference to Feigenbaum's experimental discoveries in fractal geometry, that the function of the eventual proofs of these discoveries was that of explanation, not of verification:

Lanford and other mathematicians were not trying to *validate* Feigenbaum's results any more than, say, Newton was trying to validate the discoveries of Kepler on the planetary orbits. In both cases the validity of the results was never in question. What was missing was the *explanation*. Why were the orbits ellipses? Why did they satisfy these particular relations?... There's a world of difference between validating and explaining. (p. 4, emphasis added)

Proof, then, when the results concerned are intuitively self-evident and/ or supported by convincing quasi-empirical or computer evidence, is not concerned with "making sure," but rather with "explaining why."

Furthermore, for most mathematicians, the clarification/explanation aspect of a proof is probably of *greater importance* than the aspect of verification. Halmos (quoted in Albers, 1982) noted that although the computer-assisted proof of the four-color theorem by Appel and Haken convinced him that the theorem was true, he would still personally have preferred a proof which also gave an "understanding":

I am much less likely now, after their work, to go looking for a counterexample to the four-color conjecture than I was before. To that extent, what has happened convinced me that the four-color theorem is true. I have a religious belief that some day soon, maybe six months from now, maybe sixty years from now, somebody will write a proof of the four-color theorem that will take up sixty pages in the *Pacific Journal of Mathematics*. Soon after that, perhaps six months or sixty years later, somebody will write a four-page proof, based on the concepts that in the meantime we will have developed and studied and understood. The result will belong to the grand, glorious, architectural structure of mathematics... mathematics isn't in a hurry. Efficiency is meaningless. Understanding is what counts. (pp. 239–240)

Also to Manin (1981) and Bell (1976), explanation is a criterion for a "good" proof when stating respectively, that it is "one which makes us wiser" (Manin, 1981, p. 107) and that it is expected "to convey an insight into *why* the proposition is true" (Bell, 1976, p. 24).

Critics of the amount of deductive rigor at school level often note that deduction in general (and proof in particular) is not a particularly useful heuristic device in the actual discovery of new mathematical results. This view, however, is false. There are numerous examples in the history of mathematics where new results were discovered/invented in a purely deductive manner; in fact, it is completely unlikely that some results (e.g., the non-Euclidean geometries) could ever have been chanced on merely by intuition and/or only using quasi-empirical methods. A proof that explains a result can often lead to unanticipated generalizations. To the working mathematician, therefore, proof is not merely a means of a posteriori verification, but often also a means of exploration, analysis, discovery, and invention (e.g., compare De Jager, 1990; Schoenfeld, 1986), as well as a means of systematization or communication (see de Villiers, 1990; van Asch, 1993).

AN ALTERNATIVE APPROACH TO PROOF IN GEOMETRY

Although most students who have extensively explored geometric conjectures in dynamic geometry environments usually have no further need for conviction (cf. Chazan, 1993), the author has found it relatively easy to solicit further curiosity by asking students why they think a particular result is true—to challenge them to explain it (see also Schumann & de Villiers, 1993). Students quickly admit that inductive verification merely confirms; it gives no satisfactory sense of illumination. They find it quite satisfactory, therefore, to view a deductive argument as an attempt at explanation rather than at verification.

Particularly effective as a first introduction to deductive proof appears to be to present students early on with results where the provision of proofs enables surprising further generalizations. In what follows, four examples of introductory activities that the author has used with his own mathematics education students (prospective senior primary/junior secondary teachers) are briefly discussed, and worksheets for these activites are provided in the Appendix. Some of these student teachers also have tried out similar ideas with their students with a fair amount of success in microteaching contexts or in interview situations with individual students. The author has also conducted a number of workshops with secondary teachers around these ideas, and preliminary feedback seems to indicate that such an approach to proof in dynamic geometry is meaningful (see also Koedinger, chapter 13, this volume).

Working with a Kite

Purpose. The aims of this worksheet (see Worksheet 1 in the Appendix) are (a) to allow students to discover and formulate a conjecture and (b) to guide them toward an explanation that illustrates the discovery function of proof.

Formulation. The line segments consecutively connecting the midpoints of the adjacent sides of a kite form a rectangle (see Fig. 15.5).

Deductive Explanation. A deductive analysis shows that the inscribed quadrilateral is always a rectangle, because of the *perpendicularity* of the diagonals of a kite. For example, according to an earlier discussed property of triangles, we have EF ||AC in triangle ABC and HG ||AC in triangle ADC (see Fig. 15.5a). Therefore, EF ||HG. Similarly, EH ||BD ||FG, and therefore EFGH is a parallelogram. Because BD \perp AC (property of kite) we also have, for instance, EF \perp EH, which implies that EFGH is a rectangle (a parallelogram with a right angle).

Looking Back. Notice that the property of equal adjacent sides (or an axis of symmetry through one pair of opposite angles) was not used at all. In other words, we can immediately generalize the result to a "perpendicular" quad as shown in Fig. 15.5b. Furthermore, note that the general result was not suggested by the purely empirical verification of the original conjecture. Even a systematic empirical investigation of various types of quadrilaterals would probably not have helped to discover the general case because most people would probably have restricted their investigation to the more familiar quadrilaterals such as parallelograms, rectangles, rhombuses, squares, and rectangles. (Note that from the preceding explanation we can also see that EFGH will always be a parallelogram in any quadrilateral.)



FIG. 15.5 Explaining and generalizing to perpendicular quad.

380

Working with a Triangle

Purpose. The aims of this worksheet (see Worksheet 2 in the Appendix) are (a) to allow students to discover and formulate a conjecture and (b) to guide them toward an explanation that illustrates the discovery function of proof.

Formulation. The medians of a triangle are concurrent (see Fig. 15.6).

Deductive Explanation. CD and BF are medians intersecting at O. Join A with O and extend to E on BC. We now have to show that E is the midpoint of BC. If we denote the areas of the various triangles by the following notation, area $\triangle ABC \leftrightarrow (ABC)$, we have

$$\frac{BE}{EC} = \frac{(ABE)}{(AEC)} = \frac{(OBE)}{(OCE)} = \frac{(ABE) - (OBE)}{(AEC) - (OCE)} = \frac{(ABO)}{(ACO)}$$

Similarly, we find

$$\frac{\text{CF}}{\text{FA}} = \frac{(\text{BCO})}{(\text{ABO})}$$
 and $\frac{\text{AD}}{\text{DB}} = \frac{(\text{ACO})}{(\text{BCO})}$

But AD = DB and CF = FA. Therefore, (ACO) = (BCO) and (BCO) = (ABO), which implies (ACO) = (ABO). But the areas of these two triangles are proportional to BE and EC as shown by the first equation. Thus, BE/EC = 1 implies BE = EC.

Looking Back. Looking back at the first part of the explanation, it is interesting to note that the product of the given ratios is always equal to 1, irrespective of whether D, E, and F are midpoints. For example,

$$\frac{BE}{EC} \times \frac{CF}{FA} \times \frac{AD}{DB} = \frac{(ABO)}{(ACO)} \times \frac{(BCO)}{(ABO)} \times \frac{(ACO)}{(BCO)} = 1$$

This immediately implies the following general result: If three line segments AE, BF, and CD of \triangle ABC are concurrent, then



FIG. 15.6 Explaining and generalizing to Ceva's theorem.

This interesting result, Ceva's theorem, was named after the Italian mathematician who published it in 1678. In his honor, the line segments AE, BF, and CD joining the vertices of a triangle to any given points on the opposite sides are called cevians. (Note that apart from the medians, the altitudes and angle bisectors of a triangle can be considered as cevians, if extended to meet the opposite sides.) Although it is not known exactly how he discovered this result, it is likely that he discovered it in a similar fashion as just outlined, and not merely by using construction and measurement.

Working with Equilateral Triangles

Purpose. The aims of this worksheet (see Worksheet 3 in the Appendix) are (a) to allow students to discover and formulate a conjecture and (b) to guide them toward an explanation that illustrates the discovery function of proof.

Formulation. If equilateral triangles DAB, EBC, and FCA are constructed on the sides of a right triangle ABC with $\angle B = 90^\circ$, then DC, EA, and FB are concurrent (see Fig. 15.7). If we call the observed point of concurrence O, then it looks as if the six angles formed at O are each equal to 60°. By measurement and transformation on Cabri or Sketchpad, this can easily be confirmed (see Fig. 15.7a). In other words, quadrilaterals ADBO, BECO, and CFAO must be cyclic because the exterior angles are equal to the opposite interior angles (60°). We can now use this observation to produce the following explanation. (Note that this illustrates another function of quasi-empirical testing and exploration, namely, assistance in the discovery/invention of a deductive explanation.)



FIG. 15.7 Explaining and generalizing to any triangle.

Deductive Explanation. Construct circumcircles ADB and BEC to intersect in B and O (see Fig. 15.7b). Connect O with A, B, C, D, E, and F. Then $\angle BOE = \angle BCE = 60^{\circ}$ (inscribed angles on the same chord). But $\angle BOA = 120^{\circ}$ because ADBO is cyclic. Therefore, AOE is a straight line. Similarly, DOC is a straight line. Also $\angle AOC = 360^{\circ} - (\angle BOA + \angle BOC) = 360^{\circ} - 240^{\circ} = 120^{\circ}$. Therefore, CFAO is also cyclic, and, as before, it follows that BOF is a straight line.

Looking Back. Because we did not use the property that $\angle B = 90^\circ$, it follows that this result is true for *any* triangle ABC. Again we see that the insight obtained from constructing a deductive explanation enables a further generalization. (It should be pointed out that the point O is normally called the Fermat point of a triangle.)

Working with a Quadrilateral

Purpose. The aims of this worksheet (see Worksheet 4 in the Appendix) are (a) to allow students to discover and formulate a conjecture, (b) to guide them toward two different explanations, and (c) to provide them with a Lakatosian experience by confronting them with a crossed quadrilateral shortly after they've discovered the interior angle sum of simple closed quadrilaterals.

Primitive Formulation. The sum of the angles of any quadrilateral is equal to 360° (see Fig. 15.8).

Deductive Explanation. Consider the convex and concave quadrilaterals ABCD shown in the tessellations in Fig. 15.9. In both cases, drawing diagonal BD would divide quadrilateral ABCD into two triangles ABD and BCD. (Note that the *reflexive* angle at D is the interior angle of concave quad ABCD and is therefore $360^{\circ} - 119^{\circ} = 241^{\circ}$). Because the sum of the angles of a triangle is equal to 180° , the sum of the angles of a quadrilateral is $2 \times 180^{\circ} = 360^{\circ}$.

Alternative Deductive Explanation. Consider the convex and concave quadrilaterals ABCD shown in Fig. 15.9. Imagine that you are a bug crawling from A to B, and at B you turn through the angle *p* as indicated to face in the direction of C. Continue crawling around the perimeter, turning as indicated through angles *q*, *r*, and *s* at C, D, and A, respectively, until at A you are facing in the same direction as you started. Because you are now facing in the same direction as when you started, you must have completed one revolution, 360°.

Therefore, $p + q + r + s = 360^{\circ}$. The sum of the interior angles of ABCD is given by $(180^{\circ} - p) + (180^{\circ} - q) + (180^{\circ} - r) + (180^{\circ} - s) = 720^{\circ} - (p + q + r + s) = 360^{\circ}$.

(Note that in the concave case, angle *r* is negative in relation to the other angles because it has an opposite direction of rotation. The size of the interior angle at D is therefore $180^\circ - r = 180^\circ + |r|$.)



FIG. 15.8 Discovering interior angle sum of quadrilaterals.

A Counterexample?

Construct a quadrilateral ABCD and measure its angles. Drag vertex D over side AB to obtain a figure similar to the one shown in Fig. 15.10. (On several occasions I've actually observed students using dynamic geometry software accidentally dragging polygons into crossed configurations



FIG. 15.9 Explaining exterior angle sum of convex (a) and concave (b) quadrilaterals.

such as this, something that is clearly not likely to arise or even be considered in standard paper-and-pencil work.) Is the sum of its interior angles equal to 360°? Is the figure ABCD a "quadrilateral"? How does this relate to the result formulated and explained earlier? What do we mean by "interior" angles?

Most people's first reaction to such a counterexample is one of "monster-barring," in support of the theorem that the sum of the interior angles of all quadrilaterals is 360°. We might therefore try to define a quadrilateral in such a way that figures like these are excluded. Lakatos



FIG. 15.10 The crossed quadrilateral.

(1976) describes a similar situation after the discovery of a counterexample to the Euler–Descartes theorem for polyhedra by the characters in his book:

| Delta: | But why accept the counter-example? We proved our conjec- ture—now it is a theorem. I admit that it clashes with this so- called "counter-example." One of them has to give way. But why |
|--------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | should the theorem give way, when it has been proved? It is the "criticism" that should retreat. It is fake criticism. This pair of nested cubes is not a polyhedron at all. It is a <i>monster</i> , a pathological case, not a counter-example. |
| Gamma: | Why not? A polyhedron is a solid whose surface consists of polygonal faces. And my counter-example is a solid bounded by polygonal faces. |
| Delta: | Your definition is incorrect. A polyhedron must be a <i>surface:</i> it has faces, edges, vertices, it can be deformed, stretched out on a blackboard, and has nothing to do with the concept of "solid." A <i>polyhedron is a surface consisting of a system of polygons.</i> (p. 16) |

From the preceding extract, we also see that refutation by counterexample usually depends on the meaning of the terms involved, and consequently definitions are frequently proposed and argued about. How can we define quadrilaterals? What do we mean by "interior" angles? How can we "save" the preceding theorem?

Defining

The intuitive essential meaning of a quadrilateral is that it has four sides (or line segments) and four vertices (or points). In other words, we could define a quadrilateral ABCD as a plane figure consisting of four (non-collinear) points A, B, C, and D, connected by four line segments AB, BC, CD, and DA.

According to this definition, the figure shown in Fig. 15.10 is a quadrilateral, and we refer to it as a *crossed* quadrilateral. (It also makes good sense to consider it a quadrilateral because the midpoints of its sides also form a parallelogram—see Worksheet 1 in the Appendix.) We can also refer to convex and concave quadrilaterals as simple quadrilaterals because they do not have any sides crossing each other.

Typically, students do not want to accept Fig. 15.10 as a quadrilateral. The following are responses obtained during individual interviews:

| A (Grade 11): | But the way I look at it is that these sides [AC & BD] haven't |
|---------------|-----------------------------------------------------------------|
| | been put in. This is the one side, and this is the other AB |
| | and DC are just diagonals. If you add those sides, then it'll |
| | be a complete quadrilateral, and you have four angles, and |
| | it'll equal 360°. |
| B (Grade 12): | One can't say it's a quad if the angles are not 360°. It is not |
| | 360°, therefore it is not a quadrilateral |

| C (Grade 11): | Can't you make this into a quadrilateral? Put D where |
|---------------|--------------------------------------------------------------|
| | B is, and C where D is if it lies on another type of plane, |
| | one can see that it is merely twisted. |
| D (Grade 11): | I can't give a reason, but it is not a quadrilateral I don't |
| | know why If I add these two angles [indicates angles |
| | AOD and BOC, where O is the intersection of AB and CD], |
| | then it will give you 360°. |

Students then also spontaneously tried to define a quadrilateral in such a way as to exclude crossed quadrilaterals:

| A (Grade 11): | We should say that two sides may not cross they can't in- tersect. Yes, that would be the best thing, then you can't draw something like that. |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------|
| D (Grade 11): | It must be consecutive points on a circle. How can I put it? If one goes clockwise, then the points must be consecutive, for example, A, B, C, and D. |

One way to extend the notion of "internal" angles to crossed quadrilaterals is by first analyzing and defining the notion of internal angles for convex and concave quadrilaterals and then consistently applying that definition to crossed quadrilaterals. (This is a strategy often used in mathematics when extending certain concepts beyond their original domain.)

Suppose we walk clockwise from A to B, B to C, and so on, around the perimeter of the convex quadrilateral shown in Fig. 15.11. The internal angle at each vertex will then be the angle through which the next side must be rotated clockwise (with the vertex as rotation center) to coincide with the previous side. In the same way we can now determine the internal angles of the crossed quadrilateral by walking around the perimeter as shown. This leads us to the surprising conclusion that the two reflexive angles indicated at A ($360^\circ - \angle BAD$) and D ($360^\circ - \angle ADC$) are the "internal" angles of the crossed quadrilateral ABCD.

We can now also calculate the sum of the interior angles of a crossed quadrilateral as follows: $\angle ABC + \angle BCD = acute \angle BAD + acute \angle CDA$; therefore, $\angle ABC + \angle BCD + (360^{\circ} - \angle BAD) + (360^{\circ} - \angle CDA) = 720^{\circ}$.



FIG. 15.11 Defining internal angles in quadrilaterals.

Reformulation

1. The sum of the interior angles of any simple quadrilateral is 360° . (Note that the first explanation assumes that at least one of the diagonals falls inside the quadrilateral, which makes the explanation invalid for crossed quadrilaterals. The second explanation is invalid for crossed quadrilaterals because the total turning p + q + r + s is not 360° , but 0° — the two clockwise turns at B and C are canceled out by the two anticlockwise turns at D and A).

2. The sum of the interior angles of any crossed quadrilateral is 720° (see preceeding explanation).

DISCUSSION

"Proof" was not used anywhere in the preceding activities or in their worksheets in the Appendix. Instead, the word *explanation* was used precisely to emphasize the intended function of the given deductive arguments. The problem is that the word *proof* in everyday language carries with it predominantly the idea of verification or conviction, and to use it in an introductory context would implicitly convey this meaning, even if the intended meaning is that of explanation. Tentative results with worksheets like those in the Appendix indicate that the presentation of proof in dynamic geometry as a means of explanation appears to be a viable alternative to the traditional approach.

The teacher's language is particularly crucial in this introductory phase. Instead of saying the usual, "We cannot be sure that this result is true for all possible variations, and we therefore have to (deductively) prove it to make absolutely sure," students find it much more meaningful if the teacher says: "We now know this result to be true from our extensive experimental investigation. Let us however now see if we can EXPLAIN WHY it is true in terms of other well-known geometric results, in other words, how it is a logical consequence of these other results."

It is necessary to discuss in some detail what is meant by an "explanation." For example, regular observation that the sun rises every morning clearly does not constitute an explanation; it only reconfirms the validity of the observation. To explain something, we have to explain it in terms of something else (e.g., the rotation of the earth around the polar axis). Students may need to be guided to appropriate explanations (proofs), the production of alternative explanations, and their comparison. Lack of initial participation in the actual activity of explaining (proving) has also been reported by teachers who have tried out some of these ideas at school level, and it appears that, in our experience, only after considerable concerted exposure to work of this kind do students become proficient in constructing their own explanations and critically comparing them. What is significant, however, is that, when proof is seen as explanation, substantial improvement in students' *attitudes* toward proof appears to occur.

The activities in this chapter, of course, assume that students have already, over a period of, say, 2 to 3 years, accumulated quite a large body of geometric knowledge by experimental exploration with dynamic geometry software. For example, students should already know various properties of quadrilaterals and that the line segment connecting the midpoints of two sides of a triangle is parallel to and equal to half the third side. They should also already know the area formula for triangles, properties of cyclic quadrilaterals, and the sum of the angles of a triangle.

Later we can reason "backward" to arrive at the basic axioms and definitions of geometry. This process of a posteriori axiomatization is typically used in real mathematical research: Axioms are usually not the beginning, but the end of such research.

The last section (working with a counterexample) is intended to recreate a typical Lakatosian situation where a counterexample to a result is presented after its deductive explanation (proof). To convince the students that it might be possible to consider Fig. 15.11 as a quadrilateral, it is useful to remind them of the exploration of the work with the kite and the consequent explanation, showing them that if we consecutively connect the midpoints of its sides, we also obtain a parallelogram. Only after much discussion is it possible to introduce and clarify an acceptable definition for quadrilaterals in general, making distinctions between simple closed quadrilaterals and crossed quadrilaterals, and appropriately reformulating the result. Another example that could be used as a follow-up is to consider the following and its explanation: "The opposite angles of a cyclic quadrilateral are supplementary." Again a crossed cyclic quad is the counterexample. It also should be noted that although all simple quadrilaterals tessellate, crossed quadrilaterals cannot tessellate because they overlap.

It should perhaps be pointed out that I fully agree with Hanna (1995) that Lakatos's model of heuristic refutation ought not to be taken as an allencompassing model for the philosophy of mathematics, nor for curriculum development and design in general, as there are many historical counterexamples to the process Lakatos describes. However, I believe that the Lakatosian (fallibilist) view compliments the Platonist and Formalist views of mathematics and that we should ensure that students are given activities and experiences that reflect each of these. As Davis and Hersh (1983) pointed out, none of these views are "correct" as each one is incomplete and one-sided if taken only by itself (p. 359).

SOME CONCLUDING COMMENTS

Traditionally the role and function of proof in the geometry classroom have either been completely ignored, or proof has been presented as a means of obtaining certainty (i.e., within the context of verification/ conviction). However, as pointed out in this chapter, mathematicians often construct proofs for reasons other than that of verification/conviction (cf. Hersh, 1993). The popular formalistic idea of many contemporary mathematics teachers, that conviction is a one-to-one mapping of deductive proof, should therefore be completely abandoned; conviction is not gained exclusively from proof alone, nor is the only function of proof that of verification/conviction. Not only does such an approach in a dynamic geometry environment represent intellectual dishonesty, but it does not make sense to students.

Rather than one-sidedly focusing on proof as a means of verification in geometry, the more fundamental function of explanation and discovery ought to be used to present proof as a meaningful activity. At the same time, attention should be given to the communicative aspects thereof by actually negotiating with students the criteria for acceptable evidence, explanations, and arguments. Furthermore, in mathematics, as anyone with a bit of experience will testify, the systematization function of proof comes to the fore only at a very advanced stage and should, therefore, be withheld in an introductory course to proof.

REFERENCES

- Albers, D. J. (1982). Paul Halmos: Maverick mathologist. Two-Year College Mathematics Journal, 13(4), 234–241.
- Alibert, D. (1988). Towards new customs in the classroom. For the Learning of Mathematics, 8(2), 31–35, 43.
- Appel, K., & Haken, W. (1984). The four-color problem. In D. M. Campbell & J. C. Higgens (Eds.), *Mathematics: People, problems, results* (Vol. 2, pp. 154–173). Belmont, CA: Wadsworth.
- Bell, A. W. (1976). A study of students' proof-explanations in mathematical situations. Educational Studies in Mathematics, 7, 23–40.
- Chazan, D. (1990). Quasi-empirical views of mathematics and mathematics teaching. In G. Hanna & I. Winchester (Eds.), *Creativity, thought and mathematical proof.* Toronto: Ontario Institute for Studies in Education.
- Chazan, D. (1993). High school students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24, 359–387.
- Davis, P. J. (1977). Proof, completeness, transcendentals and sampling. Journal of the Association of Computing Machines, 24, 298–310.
- Davis, P. J. & Hersh, R. (1983). The mathematical experience. London: Pelican.
- Davis, P. J. & Hersh, R. (1986). *Descartes' dream*. New York: Harcourt, Brace, & Jovanovich. De Jager, C. J. (1990). When should we use pattern? *Pythagoras*, 23, 11–14.
- de Villiers, M. D. (1990). The role and function of proof in mathematics. Pythagoras, 24, 17-24.
- Ernest, P. (1991). The philosophy of mathematics education. Bristol: Falmer.
- Gale, D. (1990). Proof as explanation. Mathematical Intelligencer, 12(1), 4.
- Gonobolin, F. N. (1975). Students' comprehension of geometric proofs. In J. W. Wilson (Ed.), Soviet studies in the psychology of learning and teaching mathematics (Vol. 12), Problems of instruction. Chicago: University of Chicago Press. (Original work published 1954)
- Grünbaum, B. (1993). Quadrangles, pentagons, and computers. Geombinatorics, 3, 4–9.
- Hanna, G. (1983). Rigorous proof in mathematics education. Toronto: Ontario Institute for Studies in Education.
- Hanna, G. (1989). More than formal proof. For the Learning of Mathematics, 9(1), 20-23.
- Hanna, G. (1995). Challenges to the importance of proof. For the Learning of Mathematics, 15(3), 42-49.
- Hersh, R. (1979). Some proposals for reviving the philosophy of mathematics. Advances in Mathematics, 31, 31–50.

- Hersh, R. (1993). Proving is convincing and explaining. Educational Studies in Mathematics, 24, 389–399.
- Horgan, J. (1993, October). The death of proof. Scientific American, 74-82.
- Jackiw, N. The Geometer's Sketchpad [Computer software]. Berkeley, CA: Key Curriculum Press.
- Kline, M. (1973). Why Johnny can't add: The failure of the new math. New York: St. Martin's.
- Kline, M. (1982). Mathematics: The loss of certainty. Oxford: Oxford University Press.
- Kline, M. (1984). The meaning of mathematics. In D. M. Campbell & J. C. Higgens (Eds.), Mathematics: People, problems, results (Vol. 2, pp. 11–18). Belmont, CA: Wadsworth.
- Laborde, J.-M., & Bellemain, F. (1994). Cabri-geometry II [Computer software]. Dallas, TX: Texas Instruments.
- Lakatos, I. (1976). Proofs and refutations. Cambridge: Cambridge University Press.
- Lakatos, I. (1978). Mathematics, science and epistemology. Cambridge: Cambridge University Press.
- Manin, Y. I. (1981). A digression on proof. Two-Year College Mathematics Journal, 12(2), 104-107.
- Pollak, H. O. (1984). The effects of technology on the mathematics curriculum. In A. I. Olivier (Ed.), *The Australian experience: Impressions of ICME 5*. Centrahil, South Africa: Association for Mathematics Education of South Africa.
- Polya, G. (1954). Mathematics and plausible reasoning: Induction and analogy in mathematics (Vol. 1). Princeton, NJ: Princeton University Press.
- Renz, P. (1981). Mathematical proof: What it is and what it ought to be. Two-Year College Mathematics Journal, 12(2), 83–103.
- Schoenfeld, A. H. (1986). On having and using geometric knowledge. In J. Hiebert (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 225–264). Mahwah: NJ: Lawrence Erlbaum Associates.
- Schumann, H., & de Villiers, M. D. (1993). Continuous variation of geometric figures: Interactive theorem finding and problems in proving. *Pythagoras*, 31, 9–20.
- van Asch, A. G. (1993). To prove, why and how? International Journal of Mathematical Education in Science and Technology, 24(2), 301–313.
- Volmink, J. D. (1990). The nature and role of proof in mathematics education. *Pythagoras*, 23, 7–10.

APPENDIX

Worksheet 1

- (a) Construct a dynamic kite using the properties of kites explored and discussed in our previous lessons.
- (b) Check to ensure that you have a dynamic kite, i.e., does it always remain a kite no matter how you transform the figure? Compare your construction(s) with those of your neighbors—is it the same or different?
- (c) Next construct the midpoints of the sides and connect the midpoints of adjacent sides to form an inscribed quadrilateral.
- (d) What do you notice about the inscribed quadrilateral formed in this way?
- (e) State your conjecture.

- (f) Grab any vertex of your kite and drag it to a new position. Does it confirm your conjecture? If not, can you modify your conjecture?
- (g) Repeat the previous step a number of times.
- (h) Is your conjecture also true when your kite is concave?
- Use the property checker of Cabri to check whether your conjecture is true in general.
- (j) State your final conclusion. Compare with your neighbors—is it the same or different?
- (k) Can you explain why it is true? (Try to explain it in terms of other well-known geometric results. Hint: construct the diagonals of your kite. What do you notice?)
- (l) Compare your explanation(s) with those of your neighbors. Do you agree or disagree with their explanations? Why? Which explanation is the most satisfactory? Why?

Worksheet 2

- (a) First construct a triangle. Next construct the midpoints of the sides.
- (b) Now connect the midpoint of each side with the opposite vertex of your triangle. These line segments are called medians. What do you notice about these medians?
- (c) State your conjecture.
- (d) Grab any vertex of your triangle and drag it to a new position. Does it confirm your conjecture? If not, can you modify your conjecture?
- (e) Repeat the previous step a number of times.
- (f) Is your conjecture also true when your triangle is obtuse, scalene, or right-angled?
- (h) Use the property checker of Cabri to check whether your conjecture is true in general.
- (i) State your final conclusion. Compare with your neighbors—is it the same or different?
- (j) Can you explain why it is true? (Try to explain it in terms of other well-known geometric results. Hint: Consider the ratios between the areas of triangles ABO and ACO, BCO and ABO, and ACO and BCO.)
- (k) Compare your explanation(s) with those of your neighbors. Do you agree or disagree with their explanations? Why? Which explanation is the most satisfactory? Why?

Worksheet 3

- (a) First construct a dynamic right triangle ABC with $\angle B = 90^{\circ}$.
- (b) Using the macro-construction facility of Cabri (or the script facility of Sketchpad), outwardly construct equilateral triangles DAB, EBC and FCA on the sides of your right triangle.

- (c) Draw DC, EA and FB. What do you notice?
- (d) State your conjecture.
- (e) Grab any vertex of your right triangle and drag it to a new position. Does it confirm your conjecture? If not, can you modify your conjecture?
- (e) Repeat the previous step a number of times.
- (f) Is your conjecture also true for any shape of your right triangle?
- (h) Use the property checker of Cabri to check whether your conjecture is true in general.
- (i) State your final conclusion. Compare with your neighbours—is it the same or different?
- (j) Can you explain why it is true? (Try to explain it in terms of other well-known geometric results. Hint: Consider the six angles at the point of intersection.)
- (k) Compare your explanation(s) with those of your neighbours. Do you agree or disagree with their explanations? Why? Which explanation is the most satisfactory? Why?

Worksheet 4

- (a) Construct a dynamic quadrilateral ABCD. Rotate the quadrilateral through around the midpoints of all its sides (give it half-turns). (Time-saving hint: In Cabri first define a macro-construction for half-turning a quadrilateral around the midpoint of one of its sides, and then use it for [a] and [b].)
- (b) Give each of the newly formed quadrilaterals a half-turn around the midpoint of its sides.
- (c) Measure the four angles of your quadrilateral ABCD, as well as the four angles around vertex C.
- (d) Carefully compare the angles around vertex C with the angles of ABCD. What do you notice?
- (e) What can you say about the sum of all the angles around vertex C? What does this say about the sum of the angles of quadrilateral ABCD?
- (f) Grab any vertex of your quadrilateral ABCD and drag it to a new position. What can you say about the sum of all the angles around vertex C? What does this say about the sum of the angles of quadrilateral ABCD?
- (g) Repeat the previous step a number of times.
- (h) Is your observation also true if ABCD is concave?
- (i) State your final conclusion. Compare with your neighbors—is it the same or different?
- (j) Can you explain why it is true? (Try to explain it in terms of other well-known geometric results. Hint: Draw a diagonal of the quadrilateral ABCD.)
- (k) Compare your explanation(s) with those of your neighbors. Do you agree or disagree with their explanations? Why? Which explanation is the most satisfactory? Why?

DESIGNING LEARNING ENVIRONMENTS FOR DEVELOPING UNDERSTANDING OF GEOMETRY AND SPACE

Edited by

Richard Lehrer University of Wisconsin–Madison

> Daniel Chazan Michigan State University



LAWRENCE ERLBAUM ASSOCIATES, PUBLISHERS Mahwah, New Jersey London Copyright © 1998 by Lawrence Erlbaum Associates, Inc. All rights reserved. No part of this book may be reproduced in any form, by photostat, microfilm, retrieval system, or any other means, without prior written permission of the publisher.

Lawrence Erlbaum Associates, Inc., Publishers 10 Industrial Avenue Mahwah, NJ 07430

Cover design by Kathryn Houghtaling Lacey

Library of Congress Cataloging-in-Publication Data

Lehrer, Richard. Designing Learning Environments for developing understanding of geometry and space / Richard Lehrer, Daniel Chazan p. cm. Includes bibliographical references and indexes. ISBN 0-8058-1948-7 (alk. paper). —ISBN 0-8058-1949-5 (pbk. : alk. paper)

1. Geometry—Study and teaching. I. Chazan, Daniel. II. Title.

QA461.L45 1998 516'.0071-dc21

97-48323 CIP

Books published by Lawrence Erlbaum Associates are printed on acid-free paper, and their bindings are chosen for strength and durability.

The final camera copy for this work was prepared by the author, and therefore the publisher takes no responsibility for consistency or correctness of typographical style.

Printed in the United States of America 10 9 8 7 6 5 4 3 2 1