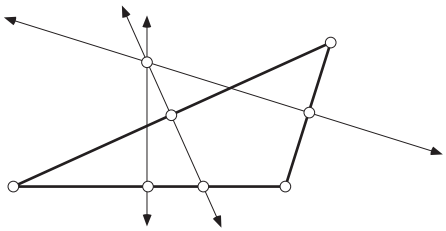
Name(s):

# Triangle Altitudes

You may have previously observed and proved that the *perpendicular bisectors* of the sides of any triangle always intersect in the same point. In this activity, you will make some conjectures about the *altitudes* of a triangle.



## Sketchpad Accuracy and Certainty

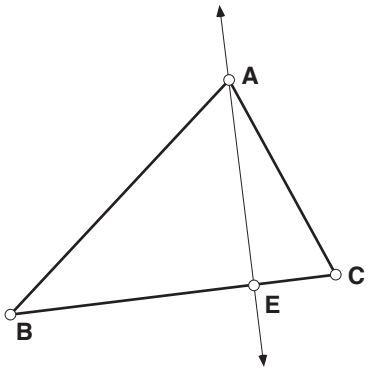
and Concurrency Conjecture activities

As you may have seen before in the Areas ~~activity~~, you should be cautious of making a judgment solely on the basis of visual appearance. Even though Sketchpad is very accurate and a powerful tool for visualization, you must still be very careful not to make false conjectures. To ensure that all measurements are set to maximum accuracy, you must look at extreme cases and where possible make use of the enlargement or animation facilities of Sketchpad to check the validity of conjectures. Ultimately, however, only a correct logical explanation/proof can assure absolute certainty.

Note that in everyday life we are often happy to say something is always true, even though we know that there may be occasional exceptions. In mathematics, however, we are only interested in conjectures that are genuinely *always* true. Mathematics is therefore different from everyday life in that absolutely no exceptions are allowed: only *one* counterexample is needed to prove a conjecture false.

## CONJECTURE

- Open the sketch **Altitudes.gsp.**
- A perpendicular from a vertex of a triangle to its opposite side is called an *altitude*. The point where an altitude meets the opposite side (or its extension) is called its *foot*. Press the buttons in your sketch to show each altitude and foot of $\Delta ABC$.




1. Drag any vertex of $\Delta ABC$. What do you notice about the altitudes of a triangle? Check to see if your observations are still true when the triangle is obtuse.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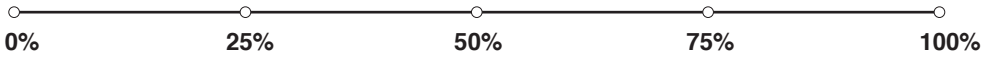
**2. Certainty:** How certain are you that your conjecture from Question 1 is always true? Record your level of certainty on the number line and explain your choice.




**CHALLENGE** If you believe your conjecture is always true, provide some examples to support your view and try to convince your partner or members of your group. Even better, support your conjecture with a logical explanation or a convincing proof. If you suspect your conjecture is not always true, try to supply counterexamples.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## PROVING

Consider the following statement by the mathematician Morris Kline (1985, 11–12):

> *Reasoning by induction and by analogy calls for recourse to observation and even experiment to obtain the facts on which to base each argument. But the senses are limited and inaccurate. Moreover, even if the facts gathered for the purposes of induction and analogy are sound, these methods do not yield unquestionable conclusions. . . .*
>
> *To avoid these sources of error, the mathematician utilizes another method of reasoning . . . in deductive reasoning the conclusion is a logically inescapable consequence of the known facts.*

**3.** Comment on how this quotation from Morris Kline is related to your work on your conjecture.

Now work in the same sketch and use the following arguments to convince yourself of the truth of your original conjecture: *The altitudes of any triangle are concurrent.*

Press the button to show some parallels constructed in your sketch.

To measure the slope of a straight object, select the object and choose **Slope** from the Measure menu.

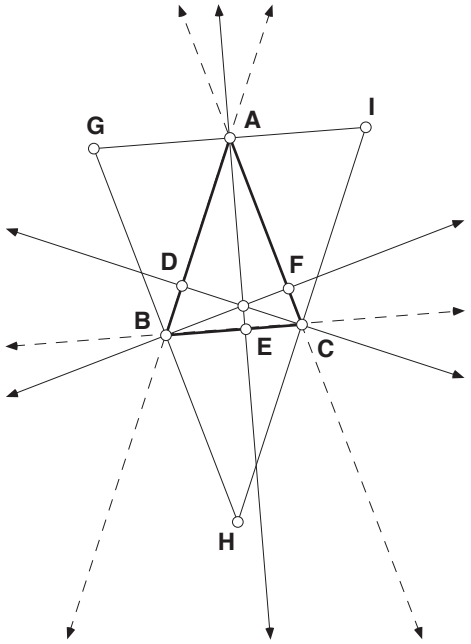
**4.** The point of concurrency of the altitudes of a triangle is called the *orthocenter*. Drag vertices of $\Delta ABC$ as you fill in these blanks:

The altitudes of $\Delta ABC$ are _____ , _____ , and _____ .

$\overline{GI}$ is parallel to _____ ,

$\overline{IH}$ is parallel to _____ , and

$\overline{GH}$ is parallel to _____ .




**5.** What can you say about quadrilateral *GBCA*? What type of quadrilateral is it? Why?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**6.** From Question 5, what can you therefore say about $\overline{GA}$ and $\overline{BC}$?

**7.** What can you say about quadrilateral *ABCI*? What type of quadrilateral is it? Why?

**8.** From Question 7, what can you therefore say about $\overline{AI}$ and $\overline{BC}$?

**9.** From Questions 6 and 8, what can you therefore say about $\overline{GA}$ and $\overline{AI}$?

**10.** What can you say about angles *GAE* and *IAE*? Why?

**11.** Explain why the original construction guarantees your observation from Question 10.

**12.** What type of line is $\overleftrightarrow{AE}$ in relation to $\overline{GI}$?

**13.** Can you say the same thing for line *BF* with respect to $\overline{GH}$, and for line *CD* with respect to $\overline{HI}$?

**14.** From Questions 12 and 13, what can you conclude about lines *AE*, *BF*, and *CD*? Why?

## Presenting Your Proof

Look over Questions 5–14. Now write a proof of your conjecture from Question 1 in your own words. You may include a demonstration sketch to support and explain your proof.


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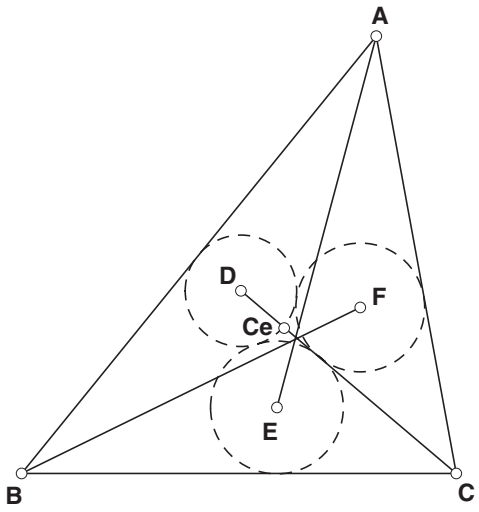


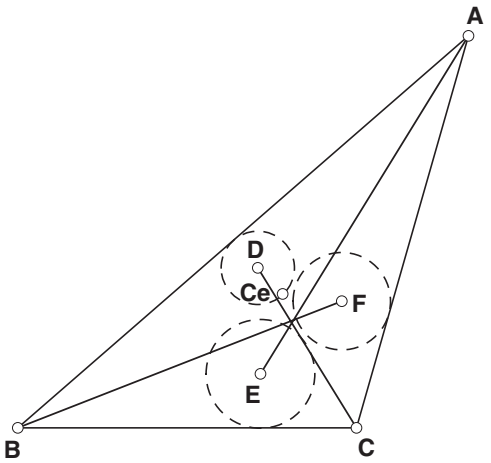


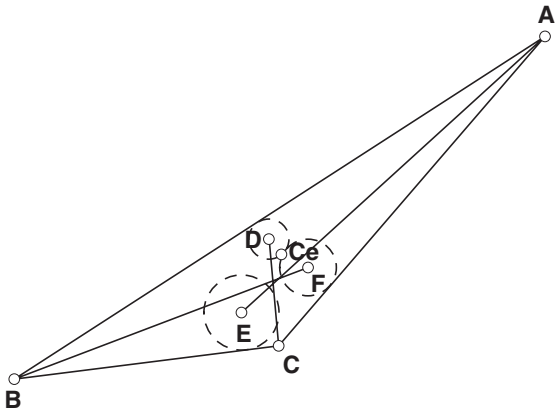


## TRIANGLE ALTITUDES (PAGE 86)

This worksheet follows the Concurrency activity and explicitly focuses on the verification function of proof.

**Prerequisites:** Water Supply I and II, Cyclic Quadrilateral, Concurrency, and Cyclic Quadrilateral Converse activities. Knowledge of the properties of cyclic quads (equal angles on same chord or opposite angles supplementary implies quad is cyclic) and the concurrency of the perpendicular bisectors of a triangle.

**Sketch: Altitudes.gsp.**

### CONJECTURE

1. The altitudes are always concurrent.
2. Responses may vary.

**CHALLENGE** It is important for you, as the teacher, to take a neutral stand here, or even better that of a skeptic, and not to indicate to the students that the result is indeed true. Challenge them to convince you or other skeptics in the class.

### PROVING

3. Responses may vary, but students are intended to recognize the verification function of proof in this quotation.
4. The altitudes are $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$. $\overline{GI} \parallel \overline{BC}$, $\overline{IH} \parallel \overline{AB}$, and $\overline{GH} \parallel \overline{AC}$.
5. *GBCA* is a parallelogram, since its opposite sides are parallel.
6. $GA = BC$ (opposite sides of parm).
7. *ABCI* is a parallelogram, since its opposite sides are parallel.
8. $AI = BC$ (opposite sides of parm).
9. $GA = AI$.
10. $m\angle GAE = 90° = m\angle IAE$, since $\overline{AE}$ is perpendicular to $\overline{BC}$ and $\overline{GI}$ is parallel to $\overline{BC}$.
11. $\overline{GI}$ has been constructed parallel to $\overline{BC}$.
12. $\overline{AE}$ is the perpendicular bisector of $\overline{GI}$.


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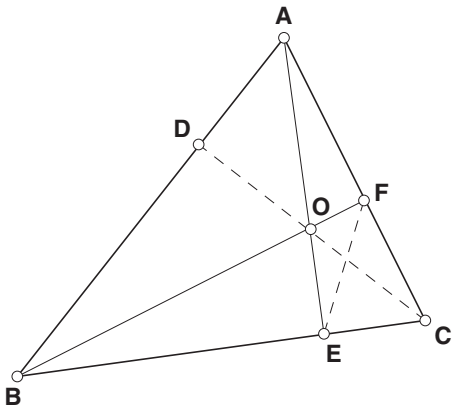
13. Yes, $\overline{BF}$ and $\overline{CD}$ are also the perpendicular bisectors of sides *GH* and *HI* of triangle *GHI*.
14. Since the perpendicular bisectors of any triangle are concurrent, $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ are concurrent. But these lines are also the altitudes of triangle *ABC*, and are therefore concurrent.

### Present Your Proof

This section provides students with the opportunity to organize the proof as a coherent whole.

### Alternative Proofs

It may also be informative for students to encounter the following well-known proof for the concurrency of the altitudes. Here, two altitudes, $\overline{AE}$ and $\overline{BF}$, are drawn, and it must now be shown that the line *CD* from the remaining vertex through their point of intersection *O* is also an altitude.




*OECF* is cyclic, since the opposite angles at *F* and *E* measure 90°. *ABEF* is also cyclic, since $m\angle AFB = 90° = m\angle AEB$ on segment *AB*. Angle *OEF* = angle *OCF* (angles on chord *OF* of *OECF*). But angle *OEF* = angle *ABF* (angles on chord *AF* of *ABEF*). Therefore, angle *OCF* = angle *ABF*, which implies that *DBEF* is cyclic (angle *DBO* = angle *DCF* on segment *DF*). Thus, $m\angle BDO = 90°$, since it is supplementary to its opposite angle *BEO* in cyclic quad *DBOE*.

= angle BFC (equal angles on chord BC of cyclic quad DBCF).

## LIGHT RAY IN A TRIANGLE (PAGE 90)

This activity follows the Triangle Altitudes activity, although it can be done independently from that activity if students already know about the concurrency of the altitudes of a triangle.

**Prerequisites:** Knowledge of properties of cyclic quads (e.g., that equal angles on same chord or opposite angles supplementary implies that quad is cyclic).

**Sketch: Light Ray.gsp.**

### CONJECTURE

1. Angle *XYB* = angle *ZYC* and angle *YZC* = angle *XZA*, but angles *AXZ* and *BXY* are not necessarily equal.
2. These two pairs of angles are equal because of the reflections occurring on sides *BC* and *AC*. For any reflection, the angle of incidence is equal to the angle of reflection.
3. They (appear to) coincide with the feet of the altitudes. (The triangle with minimum perimeter of an acute triangle is found at the feet of the altitudes.)
4. All three pairs of angles around the feet of the altitudes are equal.
5. Yes, it is also true.
6. Responses may vary.

**CHALLENGE** It is important for you, as the teacher, to take a neutral stand here, or even better that of a skeptic, and not to indicate to the students that the result is indeed true. Challenge them to convince you or other skeptics in the class.

### PROVING ANGLE MEASURES EQUAL

7. The opposite angles are supplementary ($m\angle OEC = 90° = m\angle OFC$).
8. Since $\angle OEC$ and $\angle OFC$ are both right angles, they can both be inscribed in semicircles; therefore, *OECF* is a cyclic quadrilateral.
9. $m\angle EOC = m\angle EFC$ (on chord *EC*).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