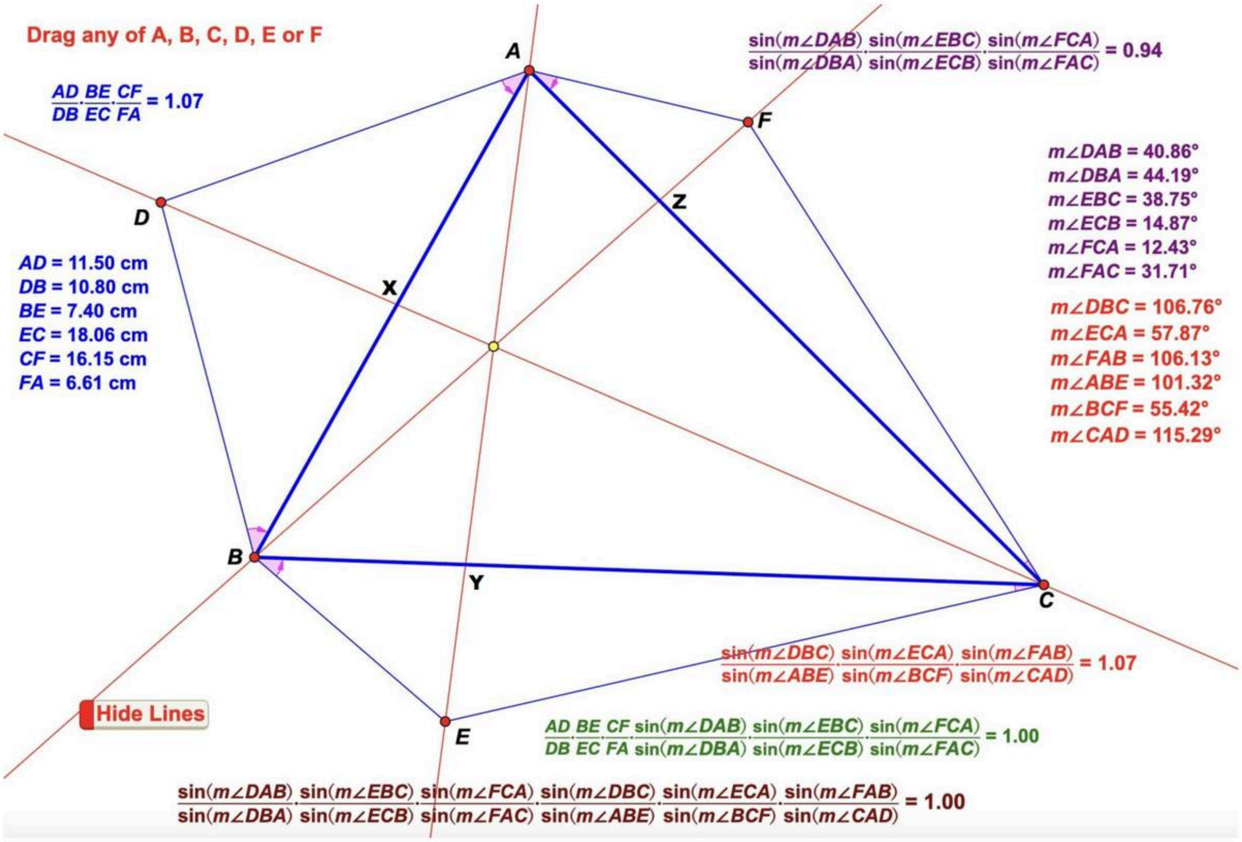
Solution by Kousik Sett (India)



Solution. Let $X = AB \cap CD$, $Y = BC \cap AE$, and $Z = CA \cap BF$. From Ceva's Theorem, we know that AX, BY, and CZ are concurrent if and only if

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = 1 \cdots (i)$$

We have

$$\frac{AX}{XB} = \frac{[DAX]}{[DBX]} = \frac{[CAX]}{[CBX]} = \frac{[DAX]+[CAX]}{[DBX]+[CBX]} = \frac{[DAC]}{[DBC]} = \frac{DA \cdot CA \cdot \sin\angle DAC}{DB \cdot BC \cdot \sin\angle DBC} \cdots (ii)$$

Similarly,

$$\frac{BY}{YC} = \frac{[EBA]}{[ECA]} = \frac{EB \cdot AB \cdot \sin\angle EBA}{EC \cdot CA \cdot \sin\angle ECA}, \cdots (iii)$$

and

$$\frac{CZ}{ZA} = \frac{[FCB]}{[FAB]} = \frac{FC \cdot BC \cdot \sin\angle FCB}{FA \cdot AB \cdot \sin\angle FAB} \cdots (iv)$$

Multiplying (ii), (iii), and (iv), and using (i), we obtain

$$\frac{DA}{DB} \cdot \frac{EB}{EC} \cdot \frac{FC}{FA} \cdot \frac{\sin\angle DAC}{\sin\angle DBC} \cdot \frac{\sin\angle EBA}{\sin\angle ECA} \cdot \frac{\sin\angle FCB}{\sin\angle FAB} = 1.$$

Finally, using the sine law on triangles ADB, BEC, and CFA, we obtain

$$\frac{\sin\angle DBA}{\sin\angle DAB} \cdot \frac{\sin\angle ECB}{\sin\angle EBC} \cdot \frac{\sin\angle FAC}{\sin\angle FCA} \cdot \frac{\sin\angle DAC}{\sin\angle DBC} \cdot \frac{\sin\angle EBA}{\sin\angle ECA} \cdot \frac{\sin\angle FCB}{\sin\angle FAB} = 1.$$

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