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International Journal of Mathematical Education in Science and Technology

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tmes20

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To cite this article: Jay Jahangiri, Ruti Segal & Moshe Stupel (2021): Angle-side properties of polygons inscribable in an ellipse*, International Journal of Mathematical Education in Science and Technology, DOI: 10.1080/0020739X.2021.1919769

To link to this article: https://doi.org/10.1080/0020739X.2021.1919769

Published online: 08 May 2021.

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY
https://doi.org/10.1080/0020739X.2021.1919769

CLASSROOM NOTE

Angle-side properties of polygons inscribable in an ellipse*

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ABSTRACT
We consider polygons that are inscribable in an ellipse and as special cases in a circle. We explore the angle-side properties of such polygons, give counterexamples for certain circumstances, and give instructions for constructing such cyclic polygons.

ARTICLE HISTORY
Received 17 March 2020

KEYWORDS
Ellipse; polygons; angle-side properties

1. Introduction

In a couple of interesting articles, Michael De Villiers (2011a, 2011b) among other results proved the following two interesting theorems on cyclic polygons, i.e. polygons with vertices upon which a circle can be circumscribed.

Cyclic polygon (De Villiers, 2011a)

A cyclic polygon has all angles equal, if and only if, the alternate sides are equal.

Cyclic 2*n*-gon (De Villiers, 2011b)

A cyclic $2n$-gon has n distinct pairs of adjacent angles equal, if and only if, one set of alternate sides are equal.

Honouring a quote by Carl Gustav Jacob Jacobi (e.g. see Davis and Hersh, 1980 or De Villiers, 2011a) that '*Man muss immer generalisieren*' (One should always generalize), here in this paper we intend to exactly do the same. We will provide intriguing extensions of the above two results to polygons inscribable in an ellipse, give counterexamples for certain cases, as well as providing an interesting method for constructing cyclic polygons with such properties.

Further exploration of classical geometrical phenomenon such as those given in De Villiers (2011a, 2011b) and those given here have magnificent application potentials (e.g. see De Villiers, 2017) and provide crucial and much needed open-ended classroom experiences for pre-service and in-service teacher education college students.

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*In memory of Walter K. Hayman (6 January 1926 – 1 January 2020) of Imperial College whose wisdom and advice the first author enjoyed during the early 1980s.

2. Main results

In the following n is a natural number and all polygons under consideration are convex and not necessarily regular unless otherwise stated.

Our first theorem explores the sum of interior angles of an inscribable polygon with an even number of vertices.

2.1. Theorem on ellipse with even diameters

For the integers $n \geq 2$, consider the ellipse with centre C so that its n diameters intersect the ellipse at the vertices $A_1, A_2, A_3, \ldots, A_n, A_{n+1} \ldots, A_{2n-1}, A_{2n}$ (Figure 1). Then for $A_{2n+1} = A_1$ and $A_{2n+2} = A_2$ we have the following two identities

(i) $\sum_{i=1}^{n} \overline{A_iA_{i+1}} = \sum_{i=1}^{n} \overline{A_{n+i}A_{n+i+1}}$
(ii) $\sum_{i=1}^{n} \angle A_iA_{i+1}A_{i+2} = \sum_{i=1}^{n} \angle A_{n+i}A_{n+i+1}A_{n+i+2}$.

Proof: Since the two triangles ΔA_iCA_{i+1} and $\Delta A_{n+i}CA_{n+i+1}$ are congruent, their corresponding sides are also congruent. So

$$\overline{A_iA_{i+1}} = \overline{A_{n+i}A_{n+i+1}}.$$

Therefore

$$\sum_{i=1}^{n} A_iA_{i+1} = \sum_{i=1}^{n} A_{n+i}A_{n+i+1}.$$

For the second part, notice that each diagonal $A_{i+1}A_{n+i+1}$ divides the corresponding interior angles $\angle A_iA_{i+1}A_{i+2}$ and $\angle A_{n+i}A_{n+i+1}A_{n+i+2}$ of the $2n$-gon $A_1A_2 \ldots A_{2n}$ into two smaller angles so that $\angle A_iA_{i+1}A_{i+2} = \angle A_iA_{i+1}C + \angle CA_{i+1}A_{i+2}$ and ■

$$\angle A_{n+i}A_{n+i+1}A_{n+i+2} = \angle A_{n+i}A_{n+i+1}C + \angle CA_{n+i+1}A_{n+i+2}.$$

Since the triangles ΔA_iCA_{i+1} and $\Delta A_{n+i}CA_{n+i+1}$ are congruent, their corresponding angles are also congruent. In particular, $\angle A_iA_{i+1}C \cong \angle A_{n+i}A_{n+i+1}C$. Similarly, the

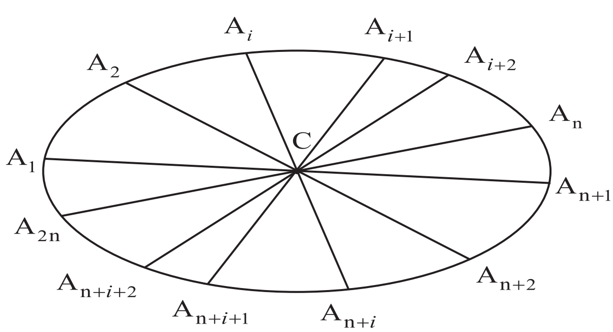

Figure 1. Ellipse with n-diameters.

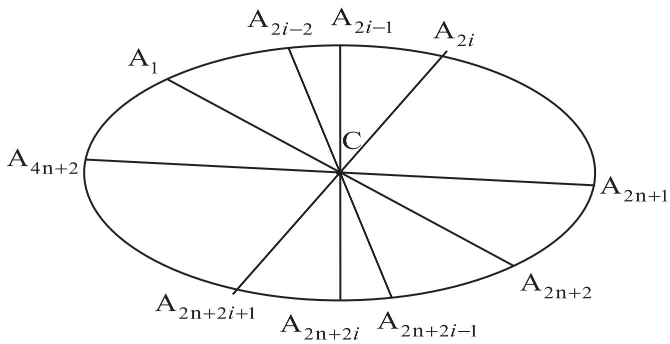

Figure 2. Ellipse with $2n+1$-diameters.

triangles $\Delta A_{i+1}CA_{i+2}$ and $\Delta A_{n+i+1}CA_{n+i+2}$ are congruent and so their corresponding angles are also congruent. In particular, $\angle CA_{i+1}A_{i+2} \cong \angle CA_{n+i+1}A_{n+i+2}$. Therefore

$$\sum_{i=1}^{n} \angle A_iA_{i+1}A_{i+2} = \sum_{i=1}^{n} \angle A_iA_{i+1}C + \sum_{i=1}^{n} \angle CA_{i+1}A_{i+2} = \sum_{i=1}^{n} \angle A_{n+i}A_{n+i+1}C + \sum_{i=1}^{n} \angle CA_{n+i+1}A_{n+i+2} = \sum_{i=1}^{n} \angle A_{n+i}A_{n+i+1}A_{n+i+2}.$$

The above Theorem 2.1 for the hexagon case is demonstrated in the following geogebra applet link https://www.geogebra.org/m/mku9xfmd

As a special case of Theorem 2.1, for the number of diameters of the ellipse being odd, we have the following interesting theorem.

2.2. Theorem on ellipse with odd diameters

For the integers $n \geq 1$, consider the ellipse with centre C so that its $2n+1$ diameters intersect the ellipse at the vertices $A_1, A_2, \ldots, A_{4n+2}$ forming the $4n+2$-gon $A_1A_2\ldots A_{4n+2}$ (Figure 2). Then for $A_{4n+2} = A_0$ and $A_{4n+3} = A_1$ we have the following two identities

(i) $\sum_{i=1}^{2n+1} \angle A_{2i-2}A_{2i-1}A_{2i} = \sum_{i=1}^{2n+1} \angle A_{2i-1}A_{2i}A_{2i+1}$
(ii) $\sum_{i=1}^{2n+1} \overline{A_{2i-1}A_{2i}} = \sum_{i=1}^{2n+1} \overline{A_{2i}A_{2i+1}}$.

Proof:

For $1 \leq i \leq n+1$, the triangles $\Delta A_{2i-1}CA_{2i}$ and $\Delta A_{2n+2i}CA_{2n+2i+1}$ are congruent. Therefore

$$\overline{A_{2i-1}A_{2i}} = \overline{A_{2n+2i}A_{2n+2i+1}}$$

and

$$\overline{A_{2i}A_{2i+1}} = \overline{A_{2n+2i+1}A_{2n+2i+2}}.$$

The sum of the sides of the $4n+2$-gon can be written as

$$\sum_{i=1}^{4n+2}\overline{A_iA_{i+1}}=\sum_{i=1}^{2n+1}\overline{A_{2i-1}A_{2i}}+\sum_{i=1}^{2n+1}\overline{A_{2n+2i}A_{2n+2i+1}}=2\sum_{i=1}^{2n+1}\overline{A_{2i-1}A_{2i}}$$

or as

$$\sum_{i=1}^{4n+2}\overline{A_iA_{i+1}}=\sum_{i=1}^{2n+1}\overline{A_{2i}A_{2i+1}}+\sum_{i=1}^{2n+1}\overline{A_{2n+2i+1}A_{2n+2i+2}}=2\sum_{i=1}^{2n+1}\overline{A_{2i}A_{2i+1}}\,.$$

■

This yields the second part of the theorem that

$$\sum_{i=1}^{2n+1}\overline{A_{2i-1}A_{2i}}=\sum_{i=1}^{2n+1}\overline{A_{2i}A_{2i+1}}.$$

For the first part, we notice that each diagonal $A_{2i-1}A_{2n+2i}$ divides the corresponding interior angles $A_{2i-2}A_{2i-1}A_{2i}$ and $A_{2n+2i-1}A_{2n+2i}A_{2n+2i+1}$ into two respective angles so that $\angle A_{2i-2}A_{2i-1}A_{2i}=\angle A_{2i-2}A_{2i-1}C+\angle CA_{2i-1}A_{2i}$ and $\angle A_{2n+2i-1}A_{2n+2i}A_{2n+2i+1}=\angle A_{2n+2i-1}A_{2n+2i}C+\angle CA_{2n+2i}A_{2n+2i+1}$.

Since the two triangles $\Delta A_{2i-2}CA_{2i-1}$ and $\Delta A_{2n+2i-1}CA_{2n+2i}$ are congruent, we have $\angle A_{2i-2}A_{2i-1}C\cong\angle A_{2n+2i-1}A_{2n+2i}C$. Similarly, the two triangles $\Delta A_{2i}A_{2i-1}C$ and $\Delta A_{2n+2i+1}A_{2n+2i}C$ are congruent yielding $\angle CA_{2i-1}A_{2i}\cong\angle CA_{2n+2i}A_{2n+2i+1}$. Therefore $\angle A_{2i-2}A_{2i-1}A_{2i}\cong\angle A_{2n+2i-1}A_{2n+2i}A_{2n+2i+1}$.

Similarly, $\angle A_{2i-1}A_{2i}A_{2i+1}\cong\angle A_{2n+2i}A_{2n+2i+1}A_{2n+2i+2}$.

Therefore, the sum of the interior angles of the $4n+2$-gon can be written as

$$\sum_{i=1}^{4n+2}\angle A_iA_{i+1}A_{i+2}=\sum_{i=1}^{2n+1}\angle A_{2i-2}A_{2i-1}A_{2i}+\sum_{i=1}^{2+1}\angle A_{2n+2i-1}A_{2n+2i}A_{2n+2i+1}=2\sum_{i=1}^{2n+1}\angle A_{2i-2}A_{2i-1}A_{2i}$$

as well as

$$\sum_{i=1}^{4n+2}\angle A_iA_{i+1}A_{i+2}=\sum_{i=1}^{2n+1}\angle A_{2i-1}A_{2i}A_{2i+1}+\sum_{i=1}^{2n+1}\angle A_{2n+2i}A_{2n+2i+1}A_{2n+2i+2}=2\sum_{i=1}^{2n+1}\angle A_{2i-1}A_{2i}A_{2i+1}$$

where $A_{4n+4}=A_2$. This yields the first part of the theorem.

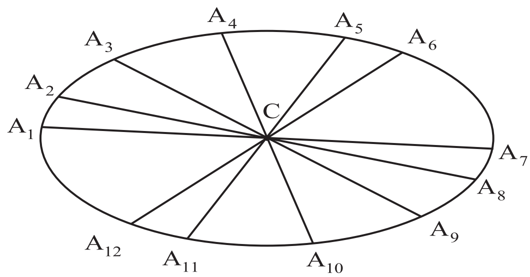

Figure 3. Ellipse with 6-diameters.

In other words, under the hypotheses of Theorem 2.2, the sum of odd-indexed interior angles of the polygon is equal to the sum of its even-indexed interior angles, that is,

$$\sum_{i=0}^{2n} \angle A_{2i}A_{2i+1}A_{2i+2} = \sum_{i=0}^{2n} \angle A_{2i+1}A_{2i+2}A_{2i+3}.$$

A similar property holds for the sum of the alternative chords of the ellipse (sides of the polygon) in Theorem 2.2.

Moreover, we note that the conclusion of Theorem 2.1 also holds for Theorem 2.2, but the reverse is not necessarily true as demonstrated in the following example.

2.3. Example.

Consider the ellipse with six diameters (Figure 3).

In Figure 3, one can easily see that $\sum_{i=1}^{6} \angle A_iA_{i+1}A_{i+2} = \sum_{i=1}^{6} \angle A_{i+6}A_{i+7}A_{i+8}$ and $\sum_{i=1}^{6} \overline{A_iA_{i+1}} = \sum_{i=1}^{6} \overline{A_{i+6}A_{i+7}}$. But neither $\sum_{i=1}^{6} \angle A_{2i}A_{2i+1}A_{2i+2}$ is necessarily equal to $\sum_{i=1}^{6} \angle A_{2i-1}A_{2i}A_{2i+1}$ nor $\sum_{i=1}^{6} \overline{A_{2i-1}A_{2i}}$ is necessarily equal to $\sum_{i=1}^{6} \overline{A_{2i}A_{2i+1}}$. Note that if the eccentricity of an ellipse is zero, then the ellipse reduces to a circle.

So, the above two Theorems 2.1 and 2.2 are also true if a circle is considered instead of an ellipse. In fact, for the circle case we can do more, that is, the vertices of the inscribed polygon can be chosen randomly not necessarily by the intersecting diagonals.

2.4. Theorem on inscribable 2n-gon

For $n \geq 2$ if the $2n$-gon $A_1A_2 \ldots A_{2n}$ is inscribable (Figure 4), then the sum of the interior angles at the odd-indexed vertices is equal to the sum of the interior angles at the even-indexed vertices, that is,

$$\sum_{i=1}^{n} \angle A_{2i-2}A_{2i-1}A_{2i} = \sum_{i=1}^{n} \angle A_{2i-1}A_{2i}A_{2i+1}.$$

Proof: Connecting the centre C to the vertices $A_1, A_2, A_3, \ldots A_{2n}$ we note that for each $1 \leq i \leq 2n$ we have $\angle A_i = \angle A_{i,1} + \angle A_{i,2}$. This forms $2n$ isosceles triangles so that $\angle A_{i,2} = \angle A_{i+1,1}$ and $\angle A_{i+1,2} = \angle A_{i+2,1}$ where $1 \leq i \leq 2n$, $\angle A_{2n+1} = \angle A_1$ and $\angle A_{2n+2} = \angle A_2$. Then, the proof is complete since ■

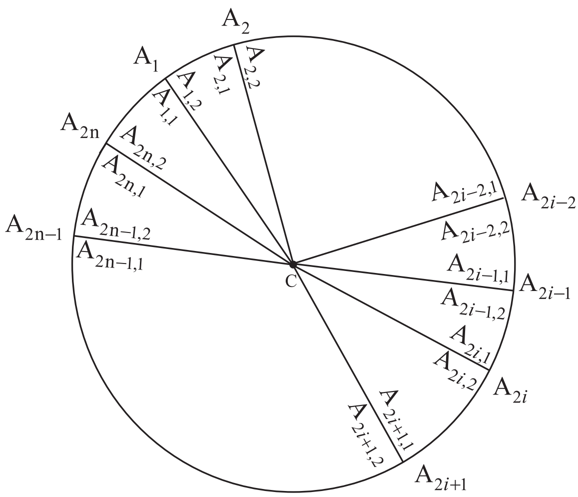

Figure 4. Inscribable 2n-gon.

$$\sum_{i=1}^{n} \angle A_{2i-1}A_{2i}A_{2i+1} = \sum_{i=1}^{n} \angle A_{2i,1} + \sum_{i=1}^{n} \angle A_{2i,2} = \sum_{i=1}^{n} \angle A_{2i-1,2} + \sum_{i=1}^{n} \angle A_{2i+1,1}$$
$$= \sum_{i=1}^{n} \angle A_{2i-1,2} + \sum_{i=1}^{n} \angle A_{2i-1,1} = \sum_{i=1}^{n} \angle A_{2i-2}A_{2i-1}A_{2i}.$$

Since the interior angle and the exterior angle at any given vertex are supplementary, it follows from Theorem 2.4 that if the $2n$-gon $A_1A_2 \dots A_{2n}$ is inscribable then the sum of the exterior angles at the odd-indexed vertices is equal to the sum of the exterior angles at the even-indexed vertices.

The above Theorem 2.4 for the hexagon and dodecagon cases are demonstrated in the following geogebra applet links https://www.geogebra.org/m/hepeenja and https://www.geogebra.org/m/utxtkzvc.

In the following theorem, we will show that the inverse of Theorem 2.4 is not true.

2.5. Theorem on non-inscribable 2n-gon

A $2n$-gon for which the sum of its odd-indexed interior angles is equal to the sum of its even-indexed interior angles is not necessarily inscribable.

Proof: Since the hexagons $A_1A_2'A_3'A_4A_5A_6$ and $A_1A_2''A_3''A_4A_5A_6$ have congruent angles to the inscribable hexagon $A_1A_2A_3A_4A_5A_6$, they satisfy the conclusion of Theorem 2.5, but obviously not its hypothesis since the hexagons $A_1A_2'A_3'A_4A_5A_6$ and $A_1A_2''A_3''A_4A_5A_6$ are not inscribable (Figure 5). ■

The above Theorem 2.5 for the hexagon case is demonstrated in the following geogebra applet link https://www.geogebra.org/m/n7bjgjra.

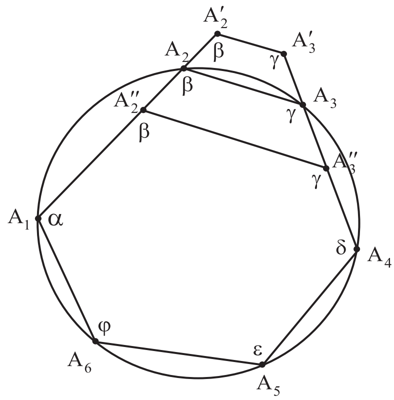

Figure 5. Non-inscribable 2*n*-gon.

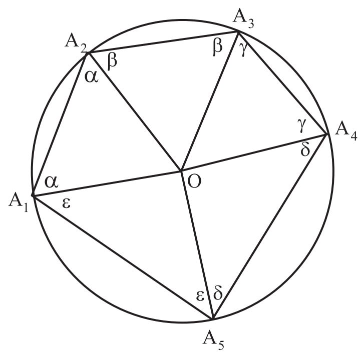

Figure 6. Inscribable 2*n*+1-gon.

We also note that Theorem 2.4 does not hold for inscribable polygons with an odd number of sides or an odd number of vertices as it is demonstrated in the following theorem.

2.6. Theorem on inscribable 2n+1-gon

For an inscribable polygon with an odd number of sides or vertices, the sum of the odd-indexed interior angles is not necessarily equal to the sum of its even-indexed interior angles.

Proof: To prove this, it suffices to give an example. ■

Consider the pentagon $A_1A_2A_3A_4A_5$ (Figure 6).

Note that $\angle A_1 + \angle A_3 + \angle A_5 = \alpha + \beta + \gamma + \delta + 2\varepsilon$.

On the other hand, the sum of the interior angles at even-indexed vertices is $\angle A_2 + \angle A_4 = \alpha + \beta + \gamma + \delta$. Since $\varepsilon > 0$, it is obvious that

$$\angle A_1 + \angle A_3 + \angle A_5 = \alpha + \beta + \gamma + \delta + 2\varepsilon > \alpha + \beta + \gamma + \delta = \angle A_2 + \angle A_4.$$

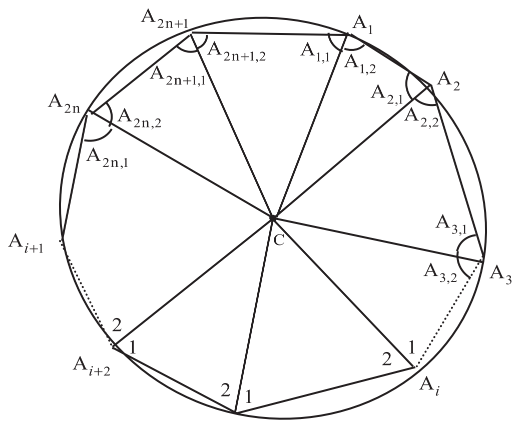

Figure 7. Regular $2n+1$-gon.

Remarkably, an inscribable $2n$-gon with congruent interior angles is not necessarily a regular polygon. An obvious example is a non-square rectangle.

Next, we examine the case for polygons with an odd number of vertices.

2.7. Theorem on regular 2n+1-gon

For integers $n \geq 1$, an inscribable $2n+1$-gon with congruent interior angles is necessarily a regular polygon.

Proof: In Figure 7, we have $\angle A_i = \angle A_{i+1}$.
Also $\angle A_{i,2} = \angle A_{i+1,1}$ where $1 \leq i \leq 2n+1$ and $\angle A_{2n+2,1} = \angle A_{1,1}$. ■

To prove that the polygon is regular, we need to show that all sides of the $2n+1-gon$ are congruent, or all the isosceles triangles are congruent.

To do so, it suffices to prove $\angle A_{i,1} = \angle A_{i,2}$ which yields that each radius CA_i bisects the interior angle $\angle A_i$. Because of the cyclic nature of the $2n+1$-*gon*, we note that $\angle A_i = \angle A_{(2n+1)+i}$. We also note that $\angle A_i = \angle A_{i,1} + \angle A_{i,2} = \angle A_{i+1} = \angle A_{i+1,1} + \angle A_{i+1,2}$. This in conjunction with $\angle A_{i,2} = \angle A_{i+1,1}$ yields $\angle A_{i,1} = \angle A_{i+1,2}$. From the identities $\angle A_{i-1,2} = \angle A_{i,1}$, $\angle A_{i+1,2} = \angle A_{i+2,1}$ and $\angle A_{i,1} = \angle A_{i+1,2}$ we obtain $\angle A_{i-1,2} = \angle A_{i,1} = \angle A_{i+1,2} = \angle A_{i+2,1}$. Therefore

$$\angle A_{i,1} = \angle A_{i+1,2} = \angle A_{i+2,1} = \angle A_{i+3,2} = \angle A_{i+4,1} = \ldots = \angle A_{i+(2n+1),2} = \angle A_{i,2}.$$

Thus the $2n+1$ isosceles triangles in Figure 8 are congruent and consequently the inscribed $2n+1$-gon is a regular polygon.

Finally, we provide an argument for the construction of a $2n$-gon with odd-even angle-sum properties regardless of being inscribable in an ellipse or a circle.

2.8. Construction of regular 2n-gon

For integers $n \geq 2$, let the $2n$-gon $B_1B_2 \ldots B_{2n}$ be constructed by the intersections of bisectors of exterior angles of a given arbitrary $2n$-gon $A_1A_2 \ldots A_{2n}$(Figure 8). Then the sum

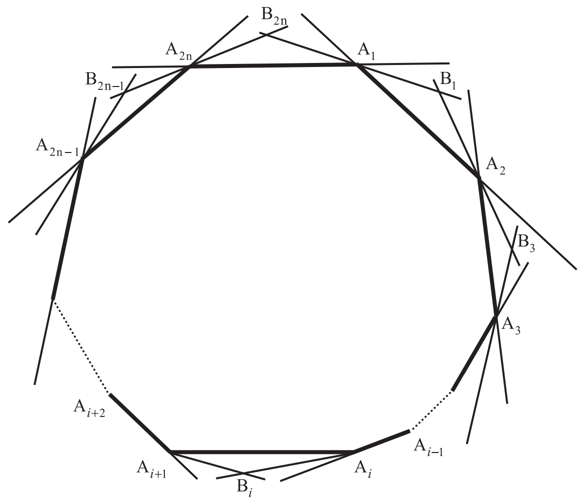

Figure 8. Construction of regular 2n-gon.

of the odd-indexed interior angles of the polygon $B_1B_2\ldots B_{2n}$ is equal to the sum of its even-indexed interior angles.

Proof: Let $\angle A_i$ denote the interior angles of the $2n-$ gon $A_1A_2\ldots A_{2n}$ and let $\angle B_i$ denote the interior angles of the $2n-$ gon $B_1B_2\ldots B_{2n}$. ■

For the typical triangle $\Delta A_iB_iA_{i+1}$ where $1 \le i \le 2n$ and $\angle A_{2n+1} = \angle A_1$ we have $\angle A_i + 2(\angle B_iA_iA_{i+1}) = 180^{\text{o}}$ and $\angle A_{i+1} + 2(\angle B_iA_{i+1}A_i) = 180^o$.

Adding the above two identities we obtain $\angle A_i + 2(\angle B_iA_iA_{i+1}) + \angle A_{i+1} + 2(\angle B_iA_{i+1}A_i) = 360^0$. On the other hand $\angle B_i + \angle B_iA_iA_{i+1} + \angle B_iA_{i+1}A_i = 180^o$. From these two equations, it follows that

$$\angle A_i + 2(\angle B_iA_iA_{i+1}) + \angle A_{i+1} + 2(\angle B_iA_{i+1}A_i) = 2(\angle B_i + \angle B_iA_iA_{i+1} + \angle B_iA_{i+1}A_i)$$

which is $\angle B_i = (\angle A_i + \angle A_{i+1}/2)$. Therefore

$$\sum_{i=1}^{n} \angle B_{2i-1} = \sum_{i=1}^{n} \frac{\angle A_{2i-1} + \angle A_{2i}}{2} = \sum_{i=1}^{2n} \angle A_i = \sum_{i=1}^{n} \frac{\angle A_{2i} + \angle A_{2i+1}}{2} = \sum_{i=1}^{n} \angle B_{2i}.$$

Although, in Theorem 2.8, the sum of the odd-indexed interior angles of the polygon $B_1B_2\ldots B_{2n}$is equal to the sum of its even-indexed interior angles, but by Theorem 2.5, the polygon $B_1B_2\ldots B_{2n}$is not necessarily inscribable. Continuous repetition of the process in Theorem 2.8 will eventually results in a 2n-gon with congruent angles.

Disclosure statement

No potential conflict of interest was reported by the authors.

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References

Davis, P. J., & Hersh, R. (1980). *The mathematical experience*. Birkhäuser.

De Villiers, M. (2011a). 95.14 Equiangular cyclic and equilateral circumscribed polygons. *The Mathematical Gazette*, *95*(532), 102–107. https://doi.org/10.1017/S0025557200002461

De Villiers, M. (2011b). Feedback: Note 95.11 equi-angular cyclic and equilateral circumscribed polygons. *The Mathematical Gazette*, *95*(533), 361.

De Villiers, M. (2017). An explanatory, transformation geometry proof of a classic treasure-hunt problem and its generalization. *International Journal of Mathematical Education in Science and Technology*, *48*(2), 260–267. https://doi.org/10.1080/0020739X.2016.1210245