Another concurrency related to the Fermat point of a triangle

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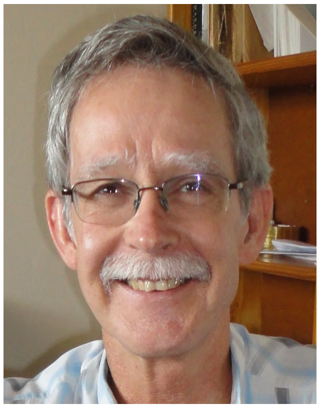

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Abstract
An interesting concurrency result related to the first Fermat (isogonic) point of a triangle was recently discovered using dynamic geometry. We provide a computer proof and an algebraic proof as well as a dynamic sketch to explore.

Introduction

In the 1600's, the French mathematician, Pierre de Fermat, posed the following intriguing problem, namely, where inside a triangle should a point be placed so that it minimizes the sum of the distances to the vertices of an acute-angled triangle?

The first one to solve it was the Italian mathematician and scientist Evangelista Torricelli (famous, in particular, for his invention of the barometer). He showed that if one constructed equilateral triangles on the sides of the triangle then the required point is located at the point of concurrency of the lines connecting the outer vertices of each equilateral triangle with the opposite vertex of the base triangle. Hence, this point of concurrency is often called the Fermat-Torricelli point.

Since that time many different proofs for the result has been given, and the concurrency result has also been further generalized, and applied in different situations (e.g. see De Villiers[4]). Free downloadable worksheets with sketches for high school learners using dynamic geometry are available in De Villiers[5], and these provide an investigative introduction to the Fermat-Torricelli point for use in the classroom.

Note that when one of the angles of the base triangle becomes greater than $120°$, then the optimal point for minimizing the sum of the distances to the vertices would be located at the angle greater than $120°$. However, the lines connecting the outer vertices of each equilateral triangle with the opposite vertex of the base triangle still remain concurrent, though the point of concurrency lies outside the base triangle. This more general point of concurrency is usually referred to as the first 'isogonic' centre of a triangle (Mackay[11]). If the equilateral triangles are constructed to the interior of the triangle, then the afore-mentioned lines are concurrent at what is called the second 'isogonic' centre.

The following interesting result related to the first Fermat (isogonic) point of a triangle was recently discovered by us using *Geometer's Sketchpad*[1]. The beauty of using dynamic geometry is that one can experimentally very quickly verify whether a result is true before trying to find a proof. In this case we proceeded by first constructing the (first) Fermat (isogonic) point of a triangle and then the second Fermat (isogonic) points of each of the three triangles into which it is divided. To our pleasant surprise, we found that the lines connecting these points to the opposite vertices of the base triangle were concurrent. While the result was new to us personally, it is quite likely that the result appears somewhere in some paper or book, and that it is already listed in the online Encyclopedia of Triangle Centers (ETC). An online dynamic sketch for readers and students to explore is available at http://dynamicmathematicslearning.com/another-concurrency-related-to-fermat.html

The Theorem

Theorem. Let D be the first Fermat (isogonic) point of $\triangle ABC$ and E, F and G be the second Fermat (isogonic) points of $\triangle ABD, \triangle BCD$ and $\triangle CAD$, respectively. Then AF, BG and CE are concurrent (see Figure 1).

Computer proof. Despite the convincing power of dragging in dynamic geometry, we duplicated the construction in *Cinderella*[2]. This was done to also check it there, since the software has a built-in 'proof-checking' facility. As shown in Figure 1, after constructing lines AF and BG and their intersection R, and next proceeding to construct line CE (line p in the sketch), immediately brought up a console stating that "R" lies on "p" (thus 'proving' the concurrency of the three lines). While it is not clear exactly how the software determines this concurrency, it appears from the manual that it is based on a technique called "randomized theorem checking" (Richter-Gebert & Kortenkamp[12] p. 48).

Presumably, the property checker of *Cinderella* is based on the mathematical theory described in Davis[3] who has pointed out that an algebraic identity can be conclusively established by a

[1] *Geometer's Sketchpad* is available for free to download from: http://dynamicmathematicslearning.com/free-download-sketchpad.html

[2] *Cinderella* is available for free to download from: https://www.cinderella.de/tiki-index.php

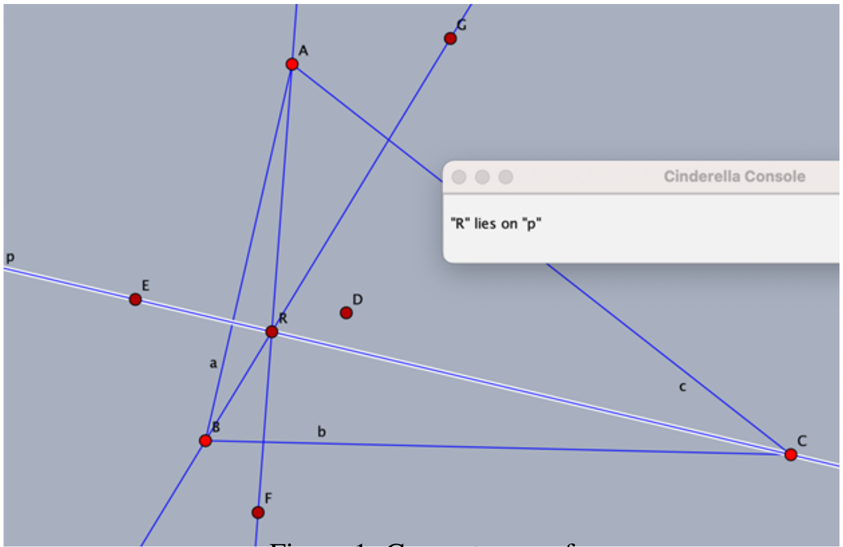

Figure 1: Computer proof

single numerical check by using algebraically independent transcendental numbers. Although computers cannot actually operate with transcendental numbers, a series of experiments selecting points at random, achieves much the same result. In other words, if experiment after experiment with randomly selected points reaffirms the same result, the probability of the result being false effectively becomes zero.

While this ‘computer proof’ with *Cinderella* provided us with further conviction of the truth of the result, we still felt the need for a proof, mainly for two reasons: 1) the given proof provides no explanation of why the result is true, and 2) the intellectual challenge of proving the result in a traditional deductive manner (compare De Villiers[6]).

The concurrency result proved to be a much harder problem than what it had seemed like initially. At first we tried using what seemed like the most natural approach, namely, the sine version of Ceva’s theorem as well as the hexagon concurrency theorem of Anghel ([1],[2]), but were unable to find a proof.

Finally, we succeeded in developing the following proof using coordinates.

□

Coordinate Proof. Place the Fermat point D of $\triangle ABC$ at the centre $(0,0)$ of the coordinate system. Let us assume (without loss of generality) that $BD \leq AD \leq CD$, with $R = BD$, $S = AD$ and $T = CD$. Therefore $R \leq S \leq T$, and $\triangle ABC$ can always be rotated or reflected so that it is placed as shown in Figure 2, with a possible permutation of A, B, C so that $R \leq S \leq T$.

Since $\angle ADB = \angle BDC = \angle CDA = 120^\circ$, three concentric regular hexagons are formed as shown in the figure, with radii R, S and T. Therefore the coordinates of the vertices of $\triangle ABC$ are $A\left(\frac{-S}{2}, \frac{\sqrt{3}S}{2}\right)$, $B\left(\frac{-R}{2}, \frac{-\sqrt{3}R}{2}\right)$ and $C(T, 0)$.

Lemma. If l_1 is the line through the points $(a_1,b_1),(c_1,d_1)$, and l_2 is the line through the points $(a_2,b_2),(c_2,d_2)$, then the x-coordinate of $l_1 \cap l_2$ (assuming l_1 is not parallel to l_2) is given by

$$\frac{[(d_1-d_2)\,a_2+c_2\,(b_2-d_1)]a_1-[(b_1-d_2)\,a_2-c_2\,(b_1-b_2)]c_1}{(a_1-c_1)\,(b_2-d_2)+(d_1-b_1)\,(a_2-c_2)}.$$

The proof is left to the reader as an exercise

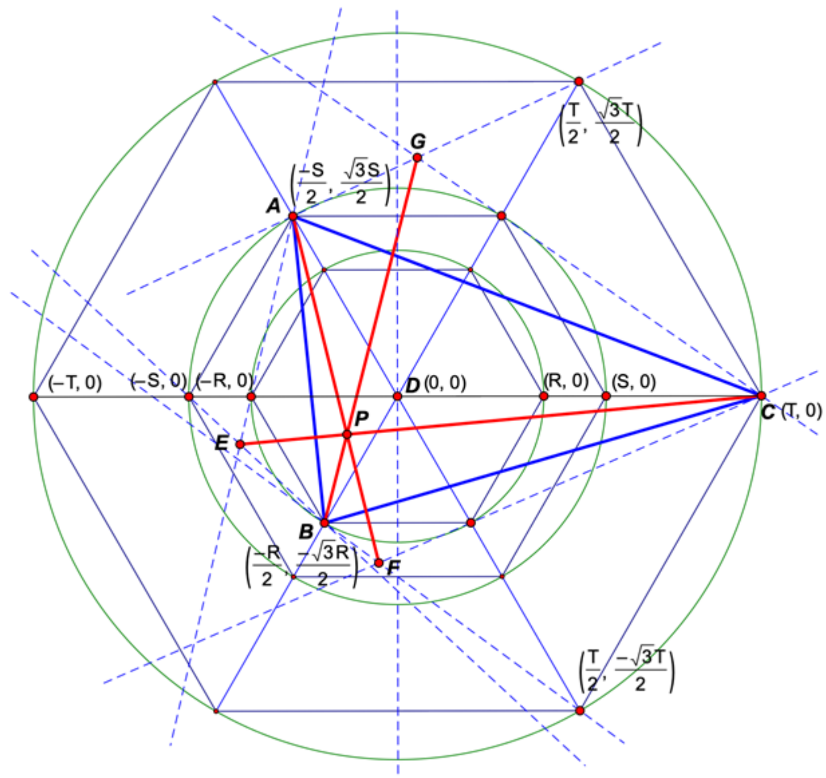

Figure 2: Coordinate proof

The y-coordinate can now similarly be determined by substituting the x-coordinate above into the equation of l_1 (or l_2).

By repeated application of this Lemma we can determine the second Fermat points G, E, F of respectively $\triangle CAD$, $\triangle ABD$ and $\triangle BCD$. For example, G is the intersection of the line through $C\,(T,0)$ and $\left(\frac{S}{2},\frac{\sqrt{3}S}{2}\right)$ and the line through $\left(\frac{T}{2},\frac{\sqrt{3}T}{2}\right)$ and $A\left(\frac{-S}{2},\frac{\sqrt{3}S}{2}\right)$.

This gives, after simplification, $G\left(\frac{ST(2S-T)}{2S^2-2ST+2T^2},\frac{\sqrt{3}ST^2}{2S^2-2ST+2T^2}\right)$.

Similarly, we get $E\left(\frac{-RS(R+S)}{2R^2-2RS+2S^2},\frac{\sqrt{3}RS(R-S)}{2R^2-2RS+2S^2}\right)$ and $F\left(\frac{RT(2R-T)}{2R^2-2RT+2T^2},\frac{-\sqrt{3}RT^2}{2R^2-2RT+2T^2}\right)$.

Now we determine the intersection points $AF \cap CE, AF \cap BG$ and $BG \cap CE$ by again using the Lemma. In each of the three intersections we get the same point

$$P\left(\frac{RST\left(2R^2S^2-T^2\left(R^2+S^2\right)\right)}{2R^3S^3+2R^3T^3+2S^3T^3+2R^2S^2T^2}, \frac{\sqrt{3}RST^3\left(R^2-S^2\right)}{2R^3S^3+2R^3T^3+2S^3T^3+2R^2S^2T^2}\right).$$

This then completes the proof of the concurrency of AF, BG and CE.

Special cases

$R = S$ ($\triangle ABC$ is isosceles) $\longrightarrow P\left(\frac{(R-T)RT}{R^2-RT+2T^2}, 0\right)$
$R = S = T$ ($\triangle ABC$ is equilateral)$\longrightarrow P(0,0) = D$.

□

It should also be mentioned that the symbolic processing software, *Maple*, was used to assist in the algebraic manipulation and simplification of the proof above.

While this proof undoubtedly satisfied our need for personal conviction and provided us with intellectual satisfaction in conquering the challenging problem facing us, it unfortunately still does not adequately explain in a simple, elegant way why the result is true. It is therefore hoped in due course that we ourselves, or perhaps others, will succeed in finding a less brute force proof of the concurrency result that is more explanatory.

Other Interesting Properties

The configuration has several other interesting mathematical properties, some of which are:

1. $\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CG}{GA} = 1$.
2. Consider Figure 3. Extend AJ to meet BI in X, and repeat the same construction on the other two sides BC and CA of $\triangle ABC$ to locate corresponding points Y and Z. Then XC, YA and ZB are concurrent. (View this construction at the URL given earlier).
3. The circumcircles of triangles AEB, BFC and CGA are concurrent. (Similarly, the circumcircles of triangles EBF, FCG and GAE are also concurrent).
4. The respective circumcenters of triangles AEB, BFC and CGA form an equilateral triangle.
5. $\angle EFG = \angle EBA + \angle ACG$, $\angle EGF = \angle EAB + \angle BCF$ and $\angle GEF = \angle GAC + \angle CBF$.

Readers are encouraged to interactively explore these properties at the URL given earlier. While the concurrency result is probably a little too hard and therefore not suitable in our opinion for possible use in a mathematics competition or training program, the five properties given above should be accessible for talented mathematics learners at different levels.

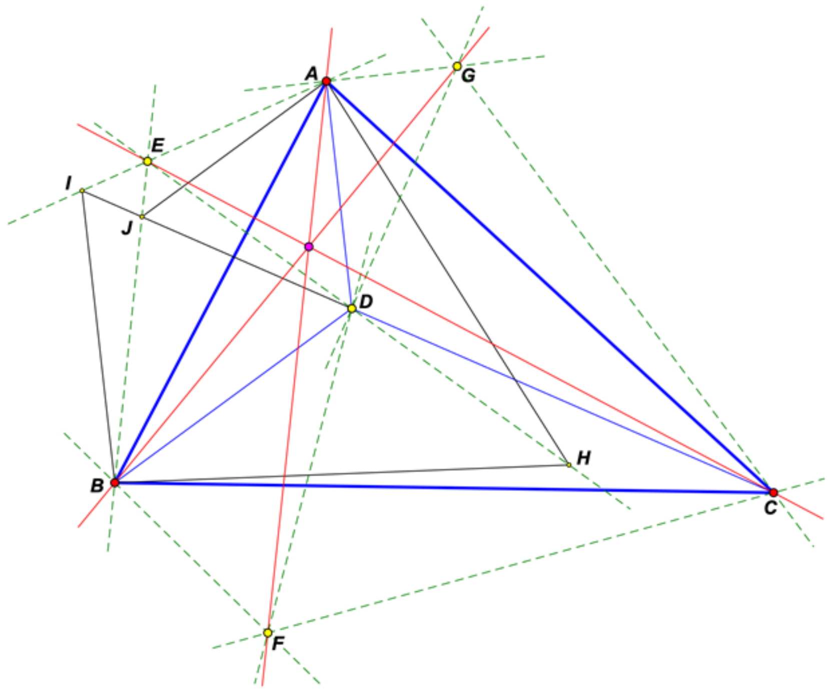

Figure 3: Proof property 1

Proof of Property 1

Consider Figure 3 which shows the relevant points as well as the constructed equilateral triangles ABH, BDI and DAJ for the location of the 2nd Fermat point E for $\triangle ABD$.

Since it is well-known that the green lines at each of E,F,G form 60° angles with each other, it follows that $AEJD$ is cyclic, since $\angle AED = 60^\circ = \angle AJD$ (angles subtended on chord AD). Therefore, if we let $\angle EAJ = x$, then $\angle EDJ = x$ (angles on chord EJ). Similarly, it follows that $EIBD$ is cyclic and that $\angle EDJ = x = \angle EBI$ (angles on chord EI). Hence, $\angle EBI = x = \angle EAJ$.

It now follows that $\angle ADE = 60^\circ - x$. But $\angle EBD = 60^\circ - x$; therefore $\angle ADE = \angle EBD$.

Apply the sine rule respectively to triangles AED and EBD to obtain the following two equations: $\frac{AE}{\sin(ADE)} = \frac{AD}{\sin(60^\circ)}$ and $\frac{ED}{\sin(EBD)} = \frac{BD}{\sin(60^\circ)}$. Divide the first equation by the second and re-arrange to obtain: $AE = \frac{ED.AD}{BD}$.

Similarly, $EB = \frac{ED.BD}{AD}$ and therefore $\frac{AE}{EB} = \frac{AD^2}{BD^2}$. In the same way can be shown that $\frac{BF}{FC} = \frac{BD^2}{CD^2}$ and $\frac{CG}{GA} = \frac{CD^2}{AD^2}$.

Therefore,

$$\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CG}{GA} = \frac{AD^2}{BD^2} \times \frac{BD^2}{CD^2} \times \frac{CD^2}{AD^2} = 1.$$

Note: While the property $\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CG}{GA} = 1$ may strongly remind one of Ceva's theorem, it is unfortunately not equivalent to it; in fact, Property 1 is neither necessary nor sufficient to prove AF, BG and CE concurrent. In general, according to the hexagon concurrency theorem of Anghel ([1],[2]), to prove AF, BG and CE concurrent, we need to prove $\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CG}{GA} = \frac{sin(EBC)}{\sin(ABF)} \times \frac{sin(FCA)}{\sin(BCG)} \times \frac{\sin(GAB)}{\sin(CAE)}$. In other words, prove that $\frac{\sin(EBC)}{\sin(ABF)} \times \frac{\sin(FCA)}{\sin(BCG)} \times \frac{\sin(GAB)}{\sin(CAE)} = 1$, but were unable to do so.

Proving the other four properties is left as an exercise to readers and students. Property 2 above follows directly from the point symmetry of the formed parallelo-hexagon $AXBYCZ$. Property 3 about the circle concurrencies is easily proven using cyclic quadrilaterals (De Villiers[7]) and Property 4 about the formed equilateral triangle is simply a variation of Napoleon's theorem (De Villiers[8]). Lastly, Property 5 follows directly from Property 1 and the application of the theorem of Egamberganov[9].

Concluding remarks

With the availability of increasingly powerful software and artificial intelligence (AI), it is perhaps prudent to ask why do we still need to deductively verify (prove) an experimentally discovered result like this when a computer or AI can produce a proof. Firstly, producing a proof can often help one to better and more deeply understand why a result is true, rather than just knowing a result is true. Secondly, while artificial intelligence and other software have already produced some impressive results, it is perhaps still cautionary to bear in mind an example given by Garaschuk[10] where five logical problems were given to ChatGPT, and while it produced plausible sounding solutions to each one, none of them were correct! Lastly, the intellectual challenge of finding a proof for oneself is what largely appeals to mathematicians, much like solving a crossword puzzle, conquering a mountain peak or running a marathon.

Web Supplement.
`http://dynamicmathematicslearning.com/another-concurrency-related-to-fermat.html`

References

[1] Anghel, N., Concurrency and Collinearity in Hexagons. *Journal for Geometry and Graphics*, 2016, Volume 20, No. 2, pp. 159–171.

[2] Anghel, N., Concurrency in Hexagons - a Trigonometric Tale. *Journal for Geometry and Graphics*, 2018, Volume 22, No. 1, pp. 21–29.

[3] Davis, P.J., Proof, completeness, transcendentals and sampling. *Journal Assoc. Comp. Machin.*, 1977, 24, pp. 298-310.

[4] De Villiers, M., From the Fermat points to the De Villiers points of a triangle. In *Proceedings of the 15 Annual AMESA Congress*, edited by Meyer, J.H. & Van Biljon, A., University of

Free State, Bloemfontein, 2009, pp. 1-8. https://dynamicmathematicslearning.com/devilliierspoints.pdf

[5] De Villiers, M., Guided Worksheets (free downloads): The Fermat-Torricelli point (pp. 108-114) https://dynamicmathematicslearning.com/Fermat-Torricelli-point.pdf and Airport Problem (pp. 115-118) https://dynamicmathematicslearning.com/Airport-Problem.pdf from *Rethinking Proof with Sketchpad* (free download), Key Curriculum Press, Emeryville, 2012a. https://www.researchgate.net/publication/375342639_Rethinking_Proof_with_Geometer%27s_Sketchpad

[6] De Villiers, M., The Role and Function of Proof with Sketchpad. Foreword from Rethinking Proof with Sketchpad (free download), Key Curriculum Press, Emeryville, 2012b.

[7] De Villiers, M., Some Circle Concurrency Theorems. *Learning and Teaching Mathematics*, 2022, No. 33, pp. 34-38

[8] De Villiers, M., A Surprise Equilateral Triangle. *Learning & Teaching Mathematics*, (In Press). Dec 2024, no. 37.

[9] Egamberganov, K., A generalization of the Napoleon's Theorem. *Mathematical Reflections*, 2017, no. 3, pp. 1-7.

[10] Garaschuk, K., Editorial. *Crux Mathematicorum*, 2023, Vol. 49, no. 1, pp. 3-4. https://cms.math.ca/wp-content/uploads/2023/02/Wholeissue_49_1-2.pdf

[11] Mackay, J.S., Isogonic Centres of a Triangle. *General Report (Association for the Improvement of Geometrical Teaching)*, The Mathematical Association, 1893, Vol. 19 (January), pp. 54-60,

[12] Richter-Gebert, J. & Kortenkamp, U., The Cinderella.2 Manual. *Technical University of Munich and the Technical University of Berlin*, 2011.

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