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Proof as a means of discovery

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CLASSROOM NOTE

Proof as a means of discovery

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ABSTRACT
This short note presents and discusses an interesting area partition result related to a parallelogram. It is, then, shown how proving the result, and understanding why the result is true based on the principle of conservation of the area of triangles with the same base and between the same parallel lines, leads to further generalizations to pentagons, hexagons, etc. The activity could be used in classroom at high school level, or in mathematics teacher education, preferably with dynamic geometry to illustrate the discovery function of proof.

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1. Introduction

The so-called discovery function of proof was introduced by De Villiers (1990) with an illustrative example of where further reflection on an 'explanatory' proof leads to a new more general result. This example illustrates that the value of a proof often goes beyond mere verification, as understanding *why* a result is true, is an added benefit not emphasized enough in the classroom. Below is an example of an elementary result that can easily be used in the classroom at high school or undergraduate level to illustrate this discovery function of proof.

Consider the following intriguing geometry result, which appeared in Jobbings (2013) and was also used as a problem in one of the Intermediate Olympiads in the UK (Richard, 2003):

Given a parallelogram $ABCD$ and arbitrary points E and F, respectively, on sides BC and CD, then as shown in Figure 1, Area $BGE +$ Area $IFD +$ Area $ECGH =$ Area $AGHI$.

It is suggested that teachers direct their students to view and investigate a dynamic version of this sketch online by selecting and dragging any of A, B, C, E or F (or for the teacher to provide them with a similar one). Go to: http://dynamicmathematicslearning.com/area-parallelogram-partition-richard-theorem.html

In the dynamic sketch, the respective areas are measured and summed, and dynamically updated as the points are dragged and the shape of the figure is changed. Empirical exploration, such as this, provides students with strong confidence in the validity of the rather surprising result, and through visual transformation, also some hint at how to go

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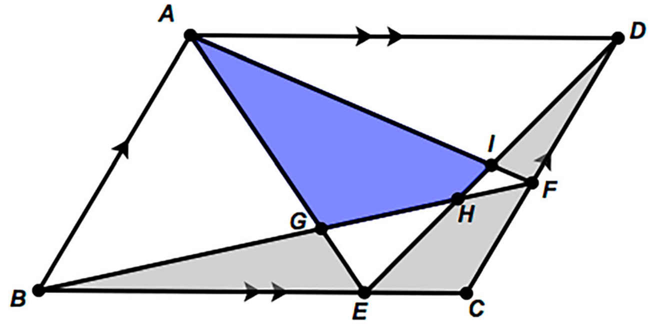

Figure 1. Area partition of parallelogram.

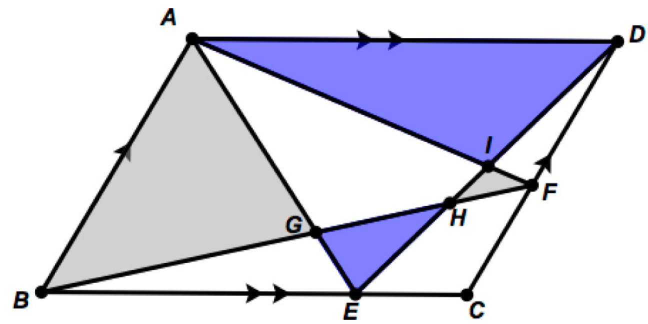

Figure 2. A simpler area partition of parallelogram.

about proving (explaining) why the result is true. In addition, a Hint button is provided in the dynamic sketch to help students focus on the conservation of the areas of the triangles with the same base and between the same parallels. The teacher might also suggest that learners label the areas of the various parts, and consider what happens with the overlap areas when J and K are dragged to A.

If students are still stuck, the teacher could suggest to them to first attempt proving the slightly simpler, auxiliary problem, shown in Figure 2, namely, the equality of areas $EGH + DIA = AGB + HIF$ in the parallelogram $ABCD$. A dynamic sketch to use for this purpose is also available at https://www.geogebra.org/m/xfs2apzs

This may help the teacher to focus students on ideas that will be relevant for the further sequence of the more complex tasks.

A proof of the more general result in Figure 1, is now given, that ideally one of the students in the class might now have found, and could then present to the whole class.

2. Explaining (proving) the parallelogram result in Figure 1

With reference to Figure 3, note that the areas of triangles JCF and KEC are, respectively, equal to the areas of triangles BCF and DEC, since they have the same base and lie between the same parallel lines. By moving points J and K to point A, as shown in Figure 4, it's clear that area JCF + area KEC = area $AECF$, and since triangles GEH an IHF lie in $AECF$ as well as in triangles BCF and DEC, the result follows. For instance, if we let the areas of BEH, GEH, $ECFH$, HFI, IFD and $AGHI$, respectively, be p, q, r, s, t and u, we then have area BCF + area DEC = area $AECF$ => $p+q+r+r+s+t = u+q+r+s$, which simplifies to $p+r+t = u$. This completes the proof.

3. Reflecting on the proof and generalizing to higher polygons

Looking back at the explanatory proof, it should be obvious that the result is true because of the two pairs of parallel sides that result in the respective equality of the areas of triangles JCF and KEC to those of areas of triangles BCF and DEC.

Asking students to consider generalizing this aspect to higher polygons is likely to have some students suggest first looking at a hexagon with opposite sides parallel. Students can, then, be invited to investigate the relationship between the areas of the triangles and quadrilaterals into which a (convex) hexagon with opposite sides parallel is partitioned by analogous lines drawn to points G and H, as shown in Figure 5. In this case, areas $(BIJ)+(NMF)+(LKDO) = (AILM)+(JGK)+(NOH)$. (The link provided earlier has a dynamic example that can be used).

Once again, the result indicated in the figure follows in exactly the same way as for a parallelogram, since AB // DE and CD // AF, and is left to the reader and/or the teacher/class to complete.

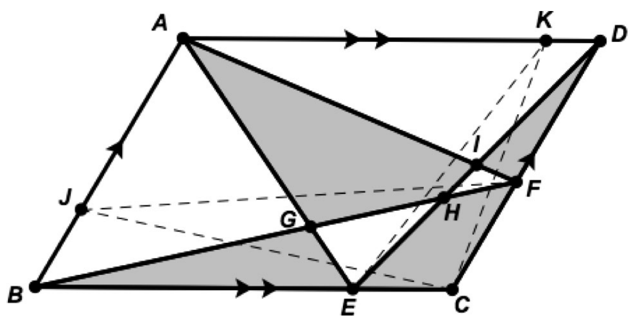

Figure 3. Explaining: Triangles with equal area.

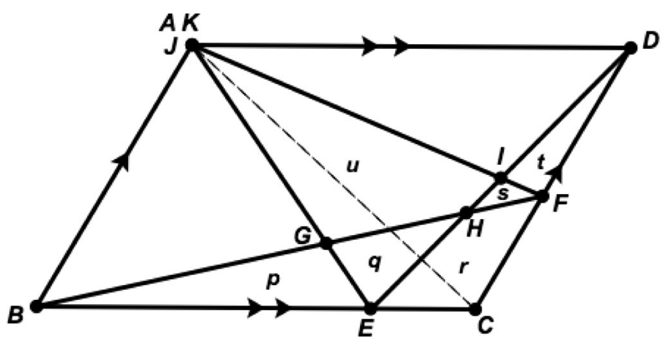

Figure 4. Explaining: Completion of proof.

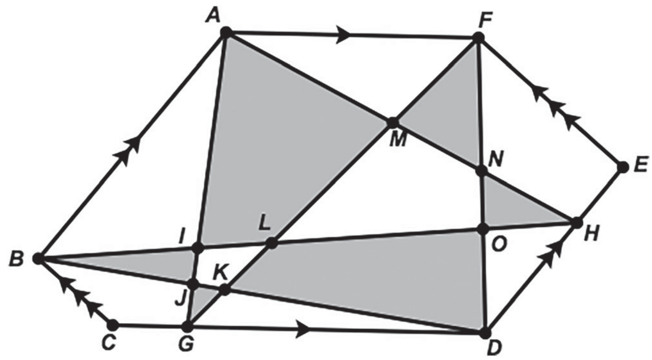

Figure 5. Area partition of hexagon with opposite sides parallel.

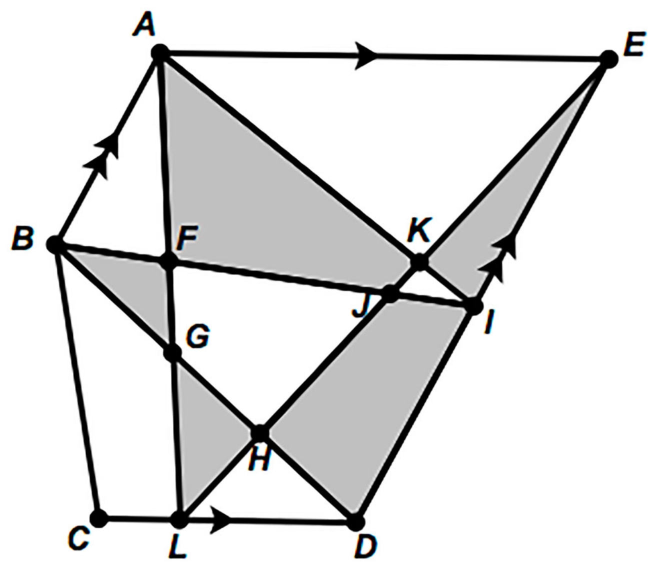

Figure 6. Area partition of pentagon with two pairs of sides parallel.

Also note that it is not necessary that BC // EF for the result to hold – so the result actually generalizes to any (convex) hexagon $ABCDEF$ with AB // DE and CD // AF. Students can now be further encouraged to generalize the result to octagons, decagons, etc.

Asking students to consider generalizing to polygons with odd numbers of vertices, like a pentagon, is likely to be first met with some scepticism, as they do not have 'opposite' sides. However, it might help to ask students to consider and investigate whether one could construct a (concave) pentagon that has some pairs of parallel lines. It is conceivable that some students may then come up with the pentagon configuration $ABCDE$ and the area result shown in Figure 6. With AE // CD, AB // ED and L and I, respectively, on CD and DE, then areas $(BFG) + (HDIJ) + (EKI) = (AFJK) + (GLH)$. (A dynamic example is also available at the link provided earlier).

Once more the result indicated in the figure holds since AB // DE and CD // AE, and can be explained (proved) in exactly the same way as before. This and generalizing it to a (convex) septagon, etc. is left to the reader.

4. Concluding remark

George Polya (1945) strongly encouraged engaging students in looking back and reflecting on proofs, and Leong et al. (2012) report on an investigation involving a student 'looking back' on a proof. In the particular instance of the result discussed here, further reflection leads to generalizing the proofs to higher polygons, and nicely illustrate the so-called discovery function of the proof mentioned earlier. This example also illustrates the point to some extent that Rav (1999) has made that it may be not so much the theorems we prove in mathematics that ultimately matter, but the proof techniques that are developed and used. In this particular case, it is being used over and over the same principle of the conservation of the area of triangles with the same base and between same parallel lines in varied polygonal configurations to prove some intriguing geometry results.

Disclosure statement

No potential conflict of interest was reported by the author.

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