'Apartheid' Definitions of Trapezia Must Fall!

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While apartheid in South Africa was finally abolished in 1994, it is sad that we often still seem to find 'apartheid' (exclusive) definitions of trapezia in school textbooks and online websites, and even being loosely used by teachers and learners in the classroom. By way of example, I recently came across the following definition of a trapezium in some South African online materials: "A trapezium is a quadrilateral with one pair of opposite sides parallel." The first problem with this definition is that it is vague and ambiguous. Do the authors mean *exactly* one pair of opposite sides parallel, or do they mean *at least* one pair of opposite sides parallel? The first interpretation of the definition would imply the exclusion of parallelograms as a special case, and it would then be an *exclusive* ('apartheid') definition. The second interpretation, however, is *inclusive* and allows us to consider a parallelogram as a special case of a trapezium.

Further down in the same online materials, however, a classification of quadrilaterals is given (as shown in Figure 1) that seems to suggest that they intended an exclusive definition for the trapezium (as well as the kite), since the classification shows no direct link between a trapezium and a parallelogram (nor between a kite and a rhombus).

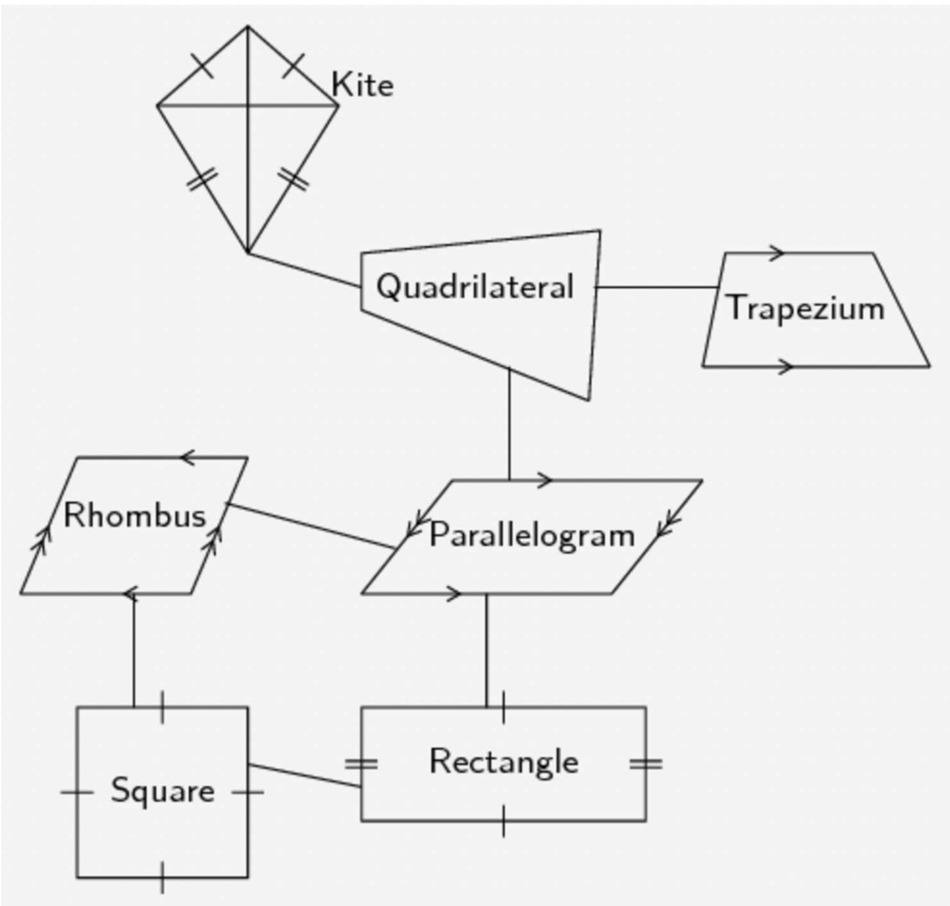

FIGURE 1: A family tree.

Two other South African websites explicitly use exclusive definitions to respectively define a trapezium as follows: “A trapezium is a quadrilateral. It has one pair of parallel sides – the other two sides are not parallel” and “a quadrilateral with only one pair of opposite sides parallel.”

While Euclid in 300 BC himself chose exclusive definitions for the quadrilaterals as can be seen in Joyce (1996) and De Villiers (2011), the mathematical community nowadays generally prefers to use inclusive, hierarchical definitions for the quadrilaterals for the following reasons (compare De Villiers, 1994):

1) it leads to more economical definitions of concepts and formulations of theorems
2) it simplifies the deductive systematization and derivation of the properties of more special concepts
3) it often provides a useful conceptual schema during problem solving
4) it sometimes suggests alternative definitions and new propositions
5) it provides a useful global perspective

It unfortunately took several centuries (gradually from about the 1600s onwards) for mathematicians to start realizing that it was more beneficial to define quadrilaterals in an inclusive, hierarchical way. Without going into too much detail, one of the major advantages of inclusive, hierarchical definitions (and the corresponding hierarchical classification) is that all the theorems one has proved for a quadrilateral then automatically apply to any of its special cases. For example, having proved that a parallelogram has diagonals that bisect each other, it is not necessary to prove it for a rectangle, rhombus or square. But using an exclusive definition for a parallelogram, one would mathematically be required to prove separately for a parallelogram, a rectangle, a rhombus and a square, in each case, that their diagonals bisect each other – so four proofs are needed instead of just one! Compared to an inclusive hierarchy of quadrilaterals, using exclusive definitions becomes ‘deductively uneconomical’ (De Villiers, 1994).

Similarly, for an inclusive hierarchical definition of a trapezium: 1) if one has proven that triangles ABE and DCE for the trapezium shown in Figure 2 have equal areas, or 2) that the centres of squares constructed on its parallel sides are collinear with the intersection of its diagonals as shown, then there is no need to prove these again for a parallelogram, since they would automatically apply. A dynamic sketch illustrating this latter result for a trapezium, as well as three others, is available at:
https://dynamicmathematicslearning.com/trapezoid.html

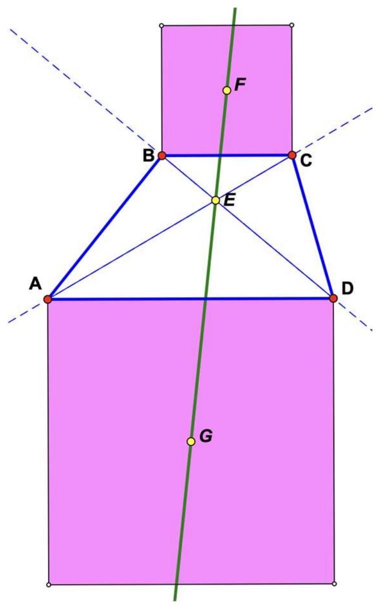

FIGURE 2: Trapezium theorem.

Perhaps the most compelling reason for defining a trapezium in an inclusive, hierarchical way is that if you construct two parallel segments AD and BC with dynamic geometry software, and connect the end points to obtain a trapezium $ABCD$ as shown in Figure 3, then that shape can be dynamically dragged into a parallelogram, rectangle, rhombus or square. It thus does not make sense in a dynamic geometry environment to choose an exclusive definition for a trapezium. A suggested learning activity for young learners involving such dynamic dragging is available online at:
https://dynamicmathematicslearning.com/trapezium.html

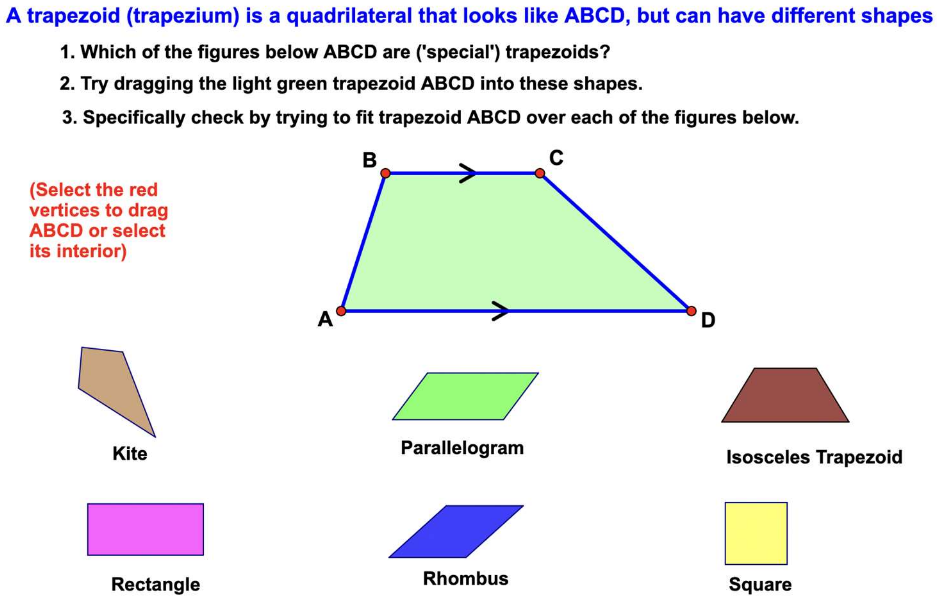

FIGURE 3: Suggested dynamically visual learning activity.

Using technology in this way, a primary school teacher can assist learners to move away from a static, fixed or rigid concept image of a trapezium, and instead develop a more robust, dynamically visual concept image that allows special cases. While dynamic activities such as this should not replace traditional geometric manipulatives such as cardboard, paper or plastic representations of various quadrilaterals, such dynamic transformations of one quadrilateral into another are essential for developing a sound conceptual framework before dealing with their formal definitions. With the concrete representations and models for quadrilaterals that are normally used in the primary or elementary school, certain static 'visual prototypes' unfortunately become fixed in the minds of learners, and eventually inhibit or prevent the development of a dynamic mental image of quadrilaterals that can transform into one another, and which would allow them to see inclusive relationships (Fujita, 2012).

In addition to initially providing young learners with similar dynamically visual experiences for the various quadrilaterals (as suggested in Figure 2 for a trapezium), learners will of course need many other carefully planned conceptual experiences. Following Van Hiele (1973), they will next need to extensively explore and learn the properties of each quadrilateral in detail, and finally they will also need to engage in constructing each quadrilateral in various different ways before they are conceptually ready to fully understand and appreciate their formal definitions (Smith, 1940; Govender & De Villiers, 2003).

Furthermore, simply providing learners with inclusive, hierarchical definitions alone will not automatically solve the problem. Unless the prerequisite conceptual framework has already been adequately formed, students may just memorize and regurgitate the given definitions for examination purposes, but their own, personal concept images of the various quadrilaterals are likely to remain static, and will invariably exclude special cases. See for example the short interview with a Grade 9 learner about his own preferred definition for a parallelogram as reported in De Villiers (1994).

While nowadays inclusive, hierarchical definitions are usually preferred whenever possible in regard to quadrilaterals, it is sometimes necessary and useful to classify and define quadrilaterals separately in partitions. For example, it is sensible and meaningful to partition the general concept of a quadrilateral into three disjunct sets, namely, convex, concave and crossed quadrilaterals.

Lastly, it should be mentioned that South Africa is not the only country where 'apartheid' definitions, specifically for a trapezium, still persist in some textbooks and online materials. While in Canada and most European countries an inclusive, hierarchical definition for a trapezium is generally preferred, in some countries like the USA and Israel, an exclusive definition for a trapezium is by far still in the majority. For example, Usiskin et al. (2008, p. 27) found in a survey of 80 textbooks in the USA that 76 used exclusive definitions for a trapezium (trapezoid) and only four used inclusive, hierarchical definitions.

Nonetheless, Usiskin et al. (2008, p. 32) write optimistically in their concluding comments of that section as follows: "The preponderance of advantages to the inclusive definition of trapezoid has caused all the articles we could find on the subject, and most college-level geometry books, to favor the inclusive definition. The inclusive definition is also the virtual unanimous choice of geometers and other mathematicians, judging from opinions expressed in the chat-room records of the Math Forum and in essays listed on the internet (Whitely 2002 and Math Forum, Trapezoid definition discussion). If definitions evolve in the 21st century as they have in prior centuries, perhaps by the end of the century the inclusive definition of trapezoid will be the one used by a majority of authors."

As we are already 25 years into this new century, let's hope that this will happen around the world in due course – probably not in my lifetime, but perhaps in the lifetime of some of our readers here? Hopefully in some small way this little article will also help contribute towards the achievement of this educational goal, and that eventually 'apartheid' definitions of trapezia will fall!

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