APPENDIX

About the speaker
SIU Man Keung obtained his BSc from the Hong Kong University and went on to earn a PhD in mathematics from Columbia University. Like the Oxford cleric in Chaucer's *The Canterbury Tales*, "and gladly would he learn, and gladly teach" for more than three decades until he retired in 2005, and is still enjoying himself in doing that after retirement. He has published some research papers in mathematics and computer science, some more papers of a general nature in history of mathematics and mathematics education, and several books in popularizing mathematics. In particular he is most interested in integrating history of mathematics with the teaching and learning of mathematics and has been participating actively in an international community of History and Pedagogy of Mathematics since the mid-1980s. He has devoted much of his time in offering a course titled *Mathematics: A Cultural Heritage* in the tradition of liberal studies for undergraduates from various Faculties of the Hong Kong University for a decade during the 2000s as well.

Example 1

The first example is a rather well-known problem in one IMO. Since I helped with the coaching of the first HK Team that was sent to take part in the 29th IMO held in Canberra in 1988, naturally I paid some special attention to the questions set in that year. Question 6 of the 29th IMO reads:

"Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that $\frac{a^2+b^2}{ab+1}$ is the square of an integer."

A slick solution to this problem, offered by a Bulgarian youngster (Emanouil Atanassov) who received a special prize for it, starts by supposing that $k=\frac{a^2+b^2}{ab+1}$ is **not** a perfect square and rewriting the expression in the form

$$a^2 - kab + b^2 = k \qquad (*) .$$

Note that for any integral solution (a,b) for (*) we have $a>0$ and $b>0$ since k is not a perfect square. Let (a,b) be an integral solution of (*) with $a>0$ and $b>0$ and $a+b$ ***smallest***. We shall produce from it another integral solution (a',b) of (*) with $a'>0$, $b>0$ and $a'+b<a+b$. This is a contradiction! [We omit the argument for arriving at such a solution (a',b).]

Slick as the proof is, it also invites a couple of queries. (1) What makes one suspect that $\frac{a^2+b^2}{ab+1}$ is the square of an integer? (2) The argument should hinge crucially upon the condition that k is not a perfect square. In the proof this condition seems to have slipped in casually so that one does not see what really goes wrong if k is *not* a perfect square. More pertinently, this proof *by contradiction* has **not explained** why $\frac{a^2+b^2}{ab+1}$ must be a perfect square, even though it **confirms** that it is so.

In contrast let us look at a much less elegant solution, which is my own attempt. When I first heard of the problem, I was on a trip in Europe and had a 'false insight' by putting $a = N^3$ and $b = N$ so that

$$a^2 + b^2 = N^2(N^4 + 1) = N^2(ab + 1) \ .$$

Under the impression that any integral solution (a, b, k) of $k = \frac{a^2+b^2}{ab+1}$ is of the form (N^3, N, N^2) I formulated a strategy of trying to deduce from $a^2 + b^2 = k(ab + 1)$ another equality

$$[a - (3b^2 - 3b + 1)]^2 + [b - 1]^2 = \{k - [2b - 1]\}\{[a - (3b^2 - 3b + 1)][b - 1] + 1\}.$$

Were I able to achieve that, I could have reduced b in steps of one until I got down to the equation $k = \frac{a^2+1}{a+1}$ for which $a = k = 1$. By reversing steps I would have solved the problem. I tried to carry out this strategy while I was travelling on a train, but to no avail. Upon returning home I could resort to systematic brute-force checking and look for some actual solutions, resulting in a (partial) list shown below.

a	**1**	**8**	**27**	30	**64**	112	**125**	**216**	240	**343**	418	**512**	…
b	**1**	**2**	**3**	8	**4**	30	**5**	**6**	27	**7**	112	**8**	…
k	**1**	**4**	**9**	4	**16**	4	**25**	**36**	9	**49**	4	**64**	…

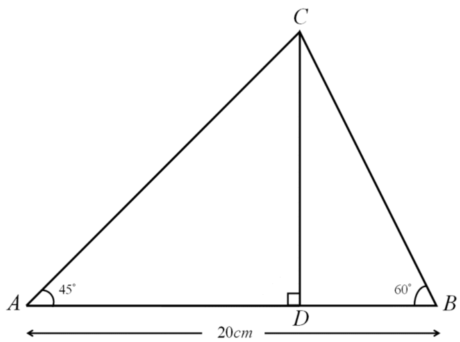
Then I saw that my ill-fated strategy was doomed to failure, because there are solutions other than those of the form (N^3, N, N^2). **However, not all was lost.** When I stared at the pattern, I noticed that for a fixed k, the solutions could be obtained recursively as (a_i, b_i, k_i) with

$$a_{i+1} = a_i k_i - b_i, \ b_{i+1} = a_i, \ k_{i+1} = k_i = k.$$

It remained to carry out the verification. Once that was done, all became clear. There is a set of 'basic solutions' of the form (N^3, N, N^2) , $N \in \{1,2,3,\ldots\}$. All other solutions are obtained from a 'basic solution' recursively as described above. In particular, $k = \frac{a^2+b^2}{ab+1}$ is the square of an integer. I feel that I understand the phenomenon much more than if I just learn from reading the slick proof.

Example 2

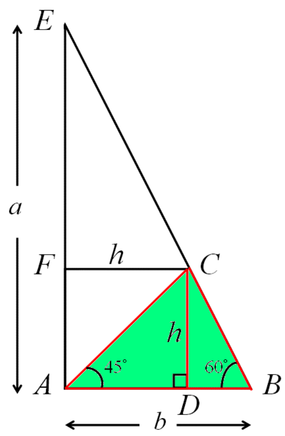
Problem 5 in the 21st Hong Kong Primary School Mathematics Competition held in May of 2010 says: Using only a ruler draw a triangle *ABC* on the A3-size paper so that *AB* is of length 20 cm., *angle BAC* is of measure 45°, and angle ABC is of measure 60°. Find the shortest distance from *C* to *AB* correct to one decimal place.



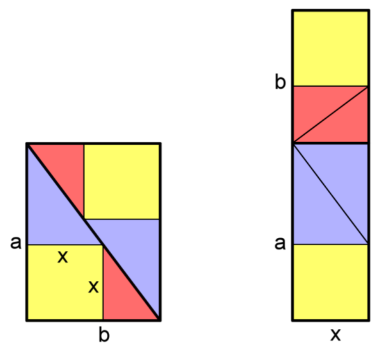
There are various ways to do this problem, which is probably originally set to see if a primary school pupil knows how to make use of paper-folding to arrive at the required triangle, then makes use of paper-folding again to get the perpendicular to the base and measures it by the ruler.

Some "early starters (who jump the gun)" tried to solve it by using trigonometry, which is not normally taught in primary school. They even knew about the Law of Sine! But they were stumped when they came to an angle of measure other than 30, 45 or 60 degrees!

Here is a clever solution which does not rely on secondary school mathematics. We extend the figure to a larger right triangle and borrow from the wisdom of ancient Chinese mathematicians.



Problem 15 in Chapter 9 of the ancient Chinese mathematical classics *Jiuzhang Suanshu* asks for the side of an inscribed square in a given right triangle, which is equal to *ab*/(*a*+*b*). [Let me demonstrate the result by the method of dissect-and-re-assemble credited to the 3rd-century Chinese mathematician LIU Hui .]



Hence, the height of the original triangle can be calculated.

This clever solution would not work if the measures of the two base angles are arbitrary, while the not-so-clever "dry" method which relies on the Law of Sine still works well.

Example 3

There is a well-known anecdote about the famous mathematician John von Neumann (1903-1957). A friend of von Neumann once gave him a problem to solve. Two cyclists A and B at a distance 20 miles apart were approaching each other, each going at a speed of 10 miles per hour. A bee flew back and forth between A and B at a speed of 15 miles per hour, starting with A and back to A after meeting B, then back to B after meeting A, and so on. By the time the two cyclists met, how far had the bee travelled? In a flash von Neumann gave the answer —— 15 miles. His friend responded by saying that von Neumann must have already known the trick so that he gave the answer so fast. His friend had in mind the slick solution to this quickie, namely, that the cyclists met after one hour so that within that one hour the bee had travelled 15 miles. To his friend's astonishment von Neumann said that he knew no trick but simply summed an infinite series!

For me this anecdote is very instructive. (1) Different people may have different ways to go about solving a mathematical problem. There is no point in forcing everybody to solve it in just the same way you solve it. Likewise, there is no point in forcing everybody to learn mathematics in just the same way you learn it. (2) Both methods of solution have their separate merits. The first method of calculating when the cyclists met is slick and captures the key point of the problem. The other method of summing an infinite series, which is slower (but not for von Neumann!) and is seemingly more cumbersome and not as clever, goes about solving the problem in a systematic manner. It indicates patience, determination, down-to-earth approach and meticulous care. Besides, it can help to consolidate some basic skills and nurture in a student a good working habit.