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A Sketchpad discovery involving triangles and quadrilaterals

A dynamic geometry sketch for the investigations discussed here is available online at:
http://dynamicmathematicslearning.com/area-inscribed-polygons.html

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The *Geometer's Sketchpad* or *Cabri* are incredible new computer programmes for exploring geometry. Explorations of the properties of triangles, quadrilaterals, circles, and other geometric configurations are very easy with *Sketchpad* or *Cabri*. It allows one's pupils or students to dynamically transform their figures with the mouse, while preserving the geometric relationships of their constructions. They are then able to examine an entire set of similar cases in a matter of seconds, leading them naturally to generalizations. *Sketchpad* or *Cabri* therefore encourages a process of discovery where pupils or students first visualise and analyze a problem, and make conjectures before attempting a logical explanation (proof) of why their observations are true.

(Sketchpad is now FREE & can be downloaded at: http://dynamicmathematicslearning.com/free-download-sketchpad.html)

What follows is an example of the use of *Sketchpad* to further explore two familiar results from high school geometry which could provide a useful enrichment exercise for more able or gifted pupils or students. An interesting result that is probably well-known to any high school teacher is the following: "If the midpoints E, F, G and H of the adjacent sides of any quadrilateral ABCD are consecutively connected, then EFGH is a parallelogram".

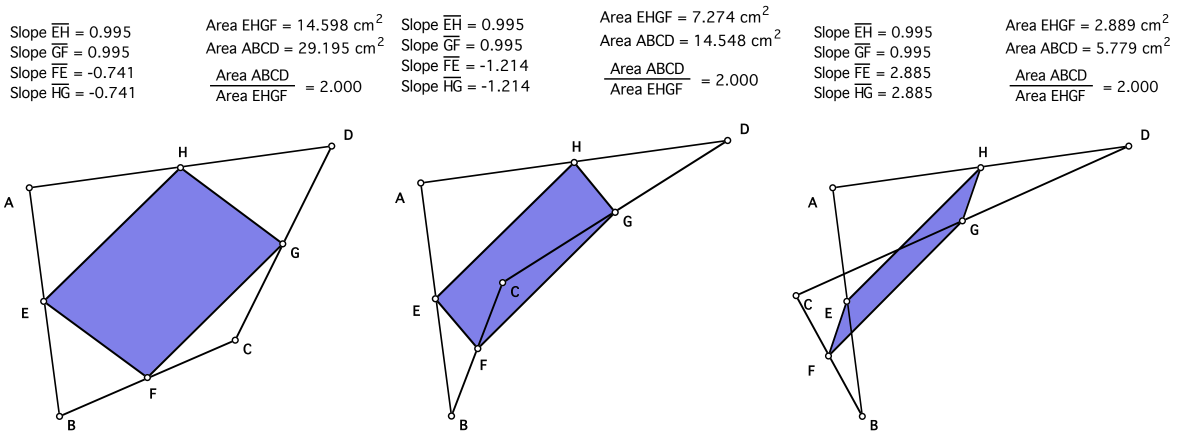

Figure 1

According to Coxeter & Greitzer (1967:53), the first known published proof of this rather simple result was only given in 1731 by Pierre Varignon, and the inscribed parallelogram is consequently often referred to as the Varignon parallelogram, and the result itself as Varignon's theorem. This result is easily demonstrated by a dynamic geometry programme like *Sketchpad* where a quadrilateral with the midpoints of its sides can easily be constructed

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and dynamically changed into any conceivable shape as shown in Figure 1. Although it is visually obvious that EFGH always remains a parallelogram as one changes the shape of ABCD, *Sketchpad* also provides a measuring facility by which one could check whether the opposite sides of EFGH are really always parallel (or equal).

What is probably less known by teachers is that the result is also true for a *crossed* quadrilateral as shown by the third figure in Figure 1. Even less well-known, is the fact also demonstrated by *Sketchpad* in Figure 1, namely, that the area of the Varignon parallelogram is exactly half of the area of the original quadrilateral, even in the case of the crossed quadrilateral!

Varignon variation

Conversely, if we start from the midpoint of AB, namely, E, and draw a line parallel to diagonal AC, it will intersect BC at its midpoint F. If we continue from F to draw a line parallel to diagonal BD, then that line would intersect CD at its midpoint G, and a line from G parallel to diagonal CA intersects DA at its midpoint H, and the line through HE is obviously parallel to diagonal DB (or alternatively, the line through H parallel to diagonal DB passes through E). What happens if we start with E not the midpoint of AB and we continue to draw lines parallel to the diagonals as above? Do we still get a parallelogram EFGH? If so, is its area still in a constant ratio to that of the original quadrilateral?

Investigation by *Sketchpad* shows that not only do we still obtain a parallelogram EFGH, but its area is also in a constant ratio to that of ABCD (see Figure 2), no matter how much one changes the shape of ABCD by dragging any of the vertices. The beauty of *Sketchpad* is of course not only that one can change the shape of ABCD, but that one can also drag E along AB to different starting positions to see how the ratio changes accordingly. Why is this general result true?

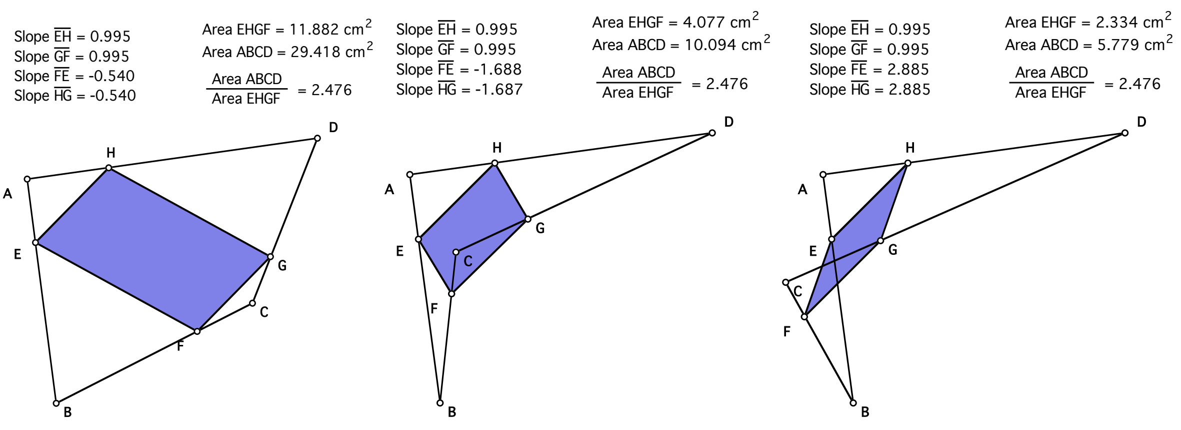

Figure 2

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Let us suppose that E divides AB into the ratio $p : q$, then since EF // AC, it follows from a well-known theorem that F also divides CB in the ratio $p : q$. Similarly, G and H respectively divide CD and AD in the ratio $p : q$. Thus, E and H respectively divide AB and AD in the same ratio and therefore EH // BD, and is EFGH a parallelogram (opposite sides are parallel to the same diagonals).

To prove the area result let us first consider the convex quadrilateral ABCD. Let us again suppose that E divides AB into the ratio $p : q$. Since triangle AEH is similar to triangle ABD, we have that $EH = \left(\frac{p}{p+q}\right)BD$ and the perpendicular from A to EH is $\frac{p}{p+q}$ times the perpendicular from A to BD. If we choose the notation (AEH) to represent the area of triangle AEH, we therefore have that (AEH) = $\left(\frac{p}{p+q}\right)^2$ (ABD). Similarly, we have (CGF) = $\left(\frac{p}{p+q}\right)^2$ (CDB), (BFE) = $\left(\frac{q}{p+q}\right)^2$ (BCA) and (DHG) = $\left(\frac{q}{p+q}\right)^2$ (DAC). Therefore, (EFGH)

= (ABCD) - (AEH) - (CGF) - (BFE) - (DHG)

= (ABCD) - $\left(\frac{p}{p+q}\right)^2$ (ABCD) - $\left(\frac{q}{p+q}\right)^2$ (ABCD)

= $\left(\frac{2pq}{(p+q)^2}\right)$ (ABCD).

This formula therefore shows that if E divides AB into a fixed ratio $p : q$, then the area of EFGH is also in a fixed ratio to that of ABCD. It is also interesting to ask for which choice of E does EFGH obtain a maximum area (or a maximum proportion of (ABCD)). Dynamically moving point E on the side AB with *Sketchpad* suggests that this maximum is obtained when E is at the midpoint. To prove that, we obviously have to find the maximum value of the fraction $\left(\frac{2pq}{(p+q)^2}\right)$. This fraction can obviously be rewritten as $\frac{2p(AB-p)}{AB^2} = \frac{-2p^2 + 2AB \bullet p}{AB^2}$, and since AB is constant, it will have a maximum when the numerator, which is a quadratic function of p, has its maximum; in other words, when $p = -\frac{2AB}{2(-2)} = \frac{1}{2}AB$.

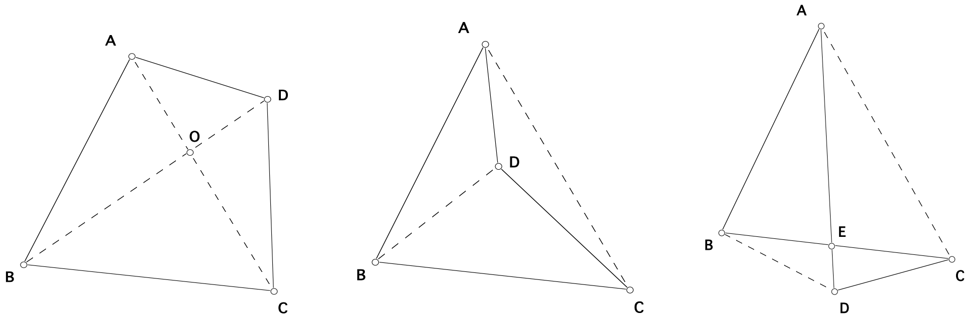

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Figure 3

In order to show that the above proof for the convex case is sufficiently general to cover the concave and crossed cases, we first have to carefully consider the meaning of "*area*" for a concave or crossed quadrilateral. It is natural to define the area of a convex quadrilateral to be the sum of the areas of the two triangles into which it is decomposed by a diagonal. For example, diagonal AC decomposes the area as follows (see Figure 3a): (ABCD) = (ABC) + (CDA).

In order to make this formula also work for the concave case shown in Figure 3b we obviously need to define (CDA)= -(ADC). In other words, we should regard the area of a triangle as being *positive* or *negative* according as its vertices are named in *counterclockwise* or *clockwise* order. For example: (ABC) = (BCA) = (CAB) = -(CBA) = -(BAC) = -(ACB).
Applying the above formula and definition of area in a crossed quadrilateral (see Figure 3c) we find that diagonal AC decomposes its area as follows:
(ABCD) = (ABC) + (CDA) = (ABC) - (ADC).

In other words, this formula forces us to regard the "*area*" of a crossed quadrilateral as the *difference* between the areas of the two small triangles ABE and EDC. [Note that diagonal BD similarly decomposes (ABCD) into (BCD) + (DAB) = -(DCB) + (DAB)]. An interesting consequence of this is that a crossed quadrilateral will have zero "*area*" if the areas of triangles ABE and EDC are equal.

We can now determine the area of EFGH in the concave case (see Figure 2b) as follows:

$$\begin{aligned}(EFGH) &= (ABCD) - (AEH) - (CGF) - (BFE) - (DHG)\\ &= (ABCD) - (AEH) + (CFG) - (BFE) - (DHG)\\ &= (ABCD) - [(AEH - (CFG)] - [(BFE) + (DHG)]\\ &= (ABCD) - \left(\frac{p}{p+q}\right)^2 (ABCD) - \left(\frac{q}{p+q}\right)^2 (ABCD)\\ &= \left(\frac{2pq}{(p+q)^2}\right) (ABCD).\end{aligned}$$

Similarly, we can also determine the area of EFGH in the crossed case (see Figure 2c):

$$\begin{aligned}(EFGH) &= (ABCD) - (AEH) - (CGF) - (BFE) - (DHG)\\ &= (ABCD) - (AEH) + (CFG) + (BEF) - (DHG)\\ &= (ABCD) - [(AEH - (CFG)] - [(DHG) - (BEF)]\\ &= (ABCD) - \left(\frac{p}{p+q}\right)^2 (ABCD) - \left(\frac{q}{p+q}\right)^2 (ABCD)\end{aligned}$$

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$$= \left(\frac{2pq}{(p+q)^2} \right) \text{(ABCD)}.$$

Triangle Variation

Another well-known theorem from high school geometry is that the midpoints of the sides of a triangle form a triangle ABC which is exactly $^1/_4$ the area of the original triangle PQR (see Figure 4a). (In fact, this triangle is not only congruent to the other three small triangles, but also similar to RQP). Conversely, if we start from the midpoint A of side PQ and draw a line parallel to QR, it will intersect PR at its midpoint B. If from B we then draw a line parallel to PQ, it will intersect RQ at its midpoint C, and therefore the line CA will be parallel to RP (or alternatively, the line through C parallel to RP passes through A).

What happens if we start with A not the midpoint of PQ and we continue to draw lines parallel to the sides as described above? Do we still get a triangle ABC? If so, is its area still in a constant ratio to that of the original triangle?

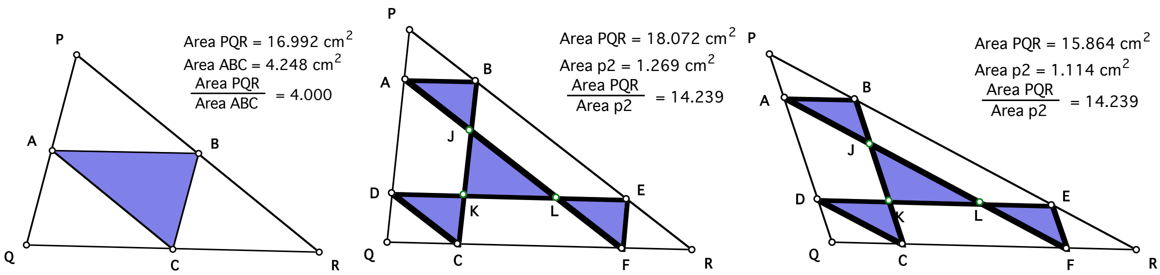

Figure 4

Investigation by *Sketchpad* shows that we do not obtain a triangle, but a crossed hexagon ABCDEF (with opposite sides parallel) (see Figure 4b-c). Nevertheless, the ratio of the area of this hexagon to that of the original triangle remains constant, no matter how much one changes the shape of PQR by dragging any of the vertices. The beauty of *Sketchpad* is of course not only that one can change the shape of PQR, but that one can also drag A along PQ to different starting positions to see how the ratio changes accordingly. Why is this general result true?

Let us suppose that A divides PQ into the ratio $p : q$, then since AB // QR, it follows from a well-known theorem that B also divides PR in the ratio $p : q$. Similarly, C, D, E and F respectively divide QR, QP, RP and RQ in the ratio $p : q$. Thus, A and F respectively divide PQ and RQ in the same ratio and therefore FA // RP, and is ABCDEF a *parallel-hexagon* (opposite sides are parallel to the same sides of PQR). (In general, the opposite sides of a hexagon ABCDEF are defined as the pairs AB and DE, BC and EF, and CD and FA).

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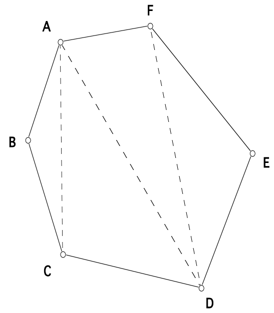

Figure 5

To prove the area result, we first need to derive a formula for the area of a crossed hexagon such as shown in Figure 4b-c. Consider a convex hexagon ABCDEF as shown in Figure 5. The area of this hexagon is decomposed by diagonal AD into quadrilaterals ABCD and DEFA, which in turn are respectively decomposed by diagonals AC and DF into triangles ABC and CDA, and DEF and FAD. This gives us the general area formula for a hexagon, namely:

(ABCDEF) = (ABCD) + (DEFA) = (ABC) + (CDA) + (DEF) + (FAD).

Again utilizing the earlier definition regarding the effect of the clockwise or counter-clockwise orientation of triangles on their areas, we can now deduce that the area of the crossed hexagon given in Figure 4b-c is given by (ABCDEF) = (ABC) + (CDA) + (DEF) + (FAD) = (ABC) + (CDA) + (DEF) - (FDA). [Note from the positions of these four triangles in Figure 4b-c, that this implies that the area of ABCDEF is equivalent to (ABJ) + (DKC) + (LEF) - (JLK)].

Let us again suppose that A divides PQ into the ratio $p : q$ (where $p \leq q$). If we let the perpendicular height from C to AB be h and the perpendicular height from P to QR be H, then from the similarity between triangles PAB and PQR follows:

$$\begin{aligned}(ABC) &= \frac{1}{2} AB \times h \\ &= \frac{1}{2}\left(\frac{p}{p+q}\right) QR \times \left(\frac{q}{p+q}\right) H \\ &= \left(\frac{pq}{(p+q)^2}\right)(PQR)\end{aligned}$$

In a similar way follows that $(DEF) = \left(\frac{pq}{(p+q)^2}\right)(PQR)$, $(CDA) = \left(\frac{(q-p)p}{(p+q)^2}\right)(PQR)$ and $(FDA) = \left(\frac{(q-p)q}{(p+q)^2}\right)(PQR)$. Substituting these into our formula (ABCDEF) = (ABC) + (CDA) + (DEF) - (FDA), we obtain:

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$$(ABCDEF) = \left(\frac{pq + pq - p^2 + pq - q^2 + pq}{(p+q)^2} \right)(PQR)$$

$$= \left(\frac{4pq - p^2 - q^2}{(p+q)^2} \right)(PQR)$$

This formula therefore clearly shows that if A divides PQ into a fixed ratio, then the area of the formed hexagon ABCD is also in a fixed ratio to the area of triangle PQR. Also note that if $p = q$, then the above fraction reduces to $^1/_2$, which means that the two triangles ABC and DEF coincide exactly, and that the area of the hexagon ABCDEF is then twice that of the midpoint triangle ABC.

It is also interesting to ask for which choice of A does ABCDEF obtain a maximum area (or a maximum proportion of (PQR)). Algebraically this means that we have to find the maximum value of the fraction $\left(\frac{4pq - p^2 - q^2}{(p+q)^2} \right)$. This fraction can obviously be rewritten as $\frac{4p(PQ-p)-(PQ-p)^2-p^2}{PQ^2} = \frac{-6p^2+6PQ \bullet p - PQ^2}{PQ^2}$, and since PQ is constant, it will have a maximum when the numerator, which is a quadratic function of p, has its maximum; in other words, when $p = -\frac{6PQ}{2(-6)} = \frac{1}{2}PQ$.

However, dynamically moving point A on the side PQ with *Sketchpad* suggests that only a local maximum for (ABCDEF) is obtained when A is at the midpoint, since higher values are obtained as A approaches one of the vertices (see Figure 6a), and that the maximum area of ABCDEF is obtained when A coincides with either P or Q (see Figure 6b). How is this possible? What is happening here? Is something wrong with our algebra or with *Sketchpad*?

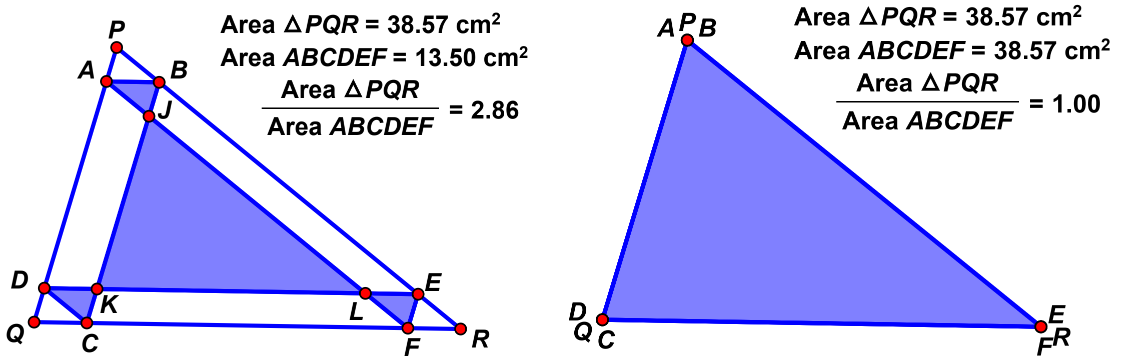

Figure 6

If we set $p = 0$ in $\frac{-6p^2+6PQ \bullet p - PQ^2}{PQ^2}$, we find (ABCDEF) = - (PQR). This negative area can easily be explained as follows: as noted earlier (ABCDEF) = (ABJ) + (DKC) + (LEF) - (JLK). So when (JLK) becomes larger than (ABJ) + (DKC) + (LEF) as in Figure 6,

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(ABCDEF) clearly becomes negative. From Figure 6, however, it is clear that *Sketchpad* only works with **positive** areas, i.e. it is taking the **absolute value** of the fraction $\left(\frac{4pq - p^2 - q^2}{(p+q)^2}\right)$.

This seems a logical convention as the area of ABCDEF can easily be expressed as a positive value for the configurations in Figure 6 by simply taking it as (JKL) - (AJB) - (DCK) - (LFE). It therefore follows that the maximum area with *Sketchpad* will be found when A coincides with either P or Q, and not at the midpoint of PQ.

It is also interesting to ask at which point does the area of ABCDEF become zero? By solving p in $-6p^2 + 6PQ \bullet p - PQ^2 = 0$ we find the symmetrical solution $\left(\frac{6 + \sqrt{12}}{12}\right)PQ = \left(\frac{3+\sqrt{3}}{6}\right)PQ$ [these are therefore the points where (JLK) becomes equal to (ABJ) + (DKC) + (LEF)].

Another interesting visual observation with *Sketchpad* is that the three triangles ABJ, DKC and LEF in Figure 4b-c appear to be congruent to each other as one moves A along PQ. This can be confirmed by measuring the corresponding sides, and is easily proved as follows. Since $AP = \left(\frac{p}{p+q}\right)PQ$ and $QD = \left(\frac{p}{p+q}\right)PQ$, we have AP = QD. But $\angle APB = \angle QDC$ since DC // PR, and $\angle DQC = \angle PAB$ since AB // QR. Therefore triangles PAB and DQC are congruent. But triangle PAB is congruent to triangle JBA since PAJB is a parallelogram. Similarly, triangle DQC is congruent to triangle CKD. Thus triangles JBA and CKD are congruent. In the same way can be shown that triangle FEL is congruent to triangles JBA and CKD.

As demonstrated here in this article, *Sketchpad* is an extremely valuable tool for exploring "*what-if?*" questions. If a conjecture turns out to be true, one immediately gets visual confirmation, which then provides one with the motivation to start looking for an explanation (proof). On the other hand, if a conjecture is false, a counter-example is easily found by dynamic transformation of the figure concerned. It is therefore no wonder that a number of school children using *Sketchpad* or other dynamic geometry software like *Cabri* have recently made some completely new, original discoveries in geometry.

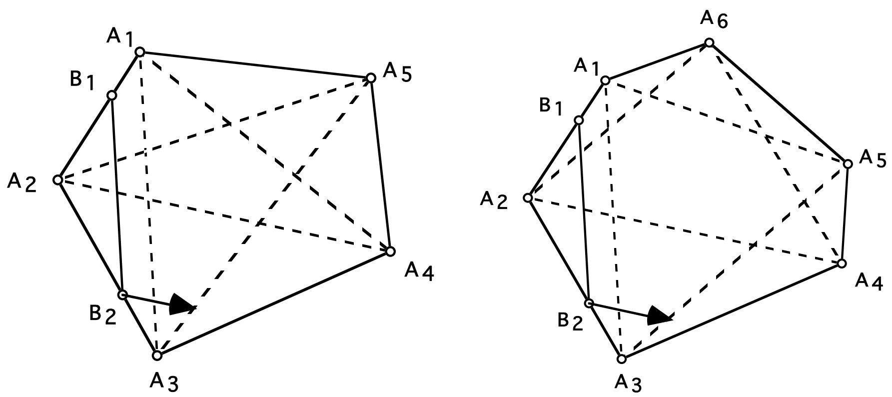

Figure 7

Lastly, the reader is left with the following further investigation from De Villiers (1996) which is ideally suited for *Sketchpad*. Consider the arbitrary pentagon and hexagon in Figure 7 and continue drawing B_jB_{j+1} parallel to the diagonals A_jA_{j+2} (j = 1; 2; 3; ...). What do you notice? Can you make a conjecture? Can you explain (prove) your conjecture? Can you generalize further? Measure the area of the formed polygon in each case and compare with the original. Is the ratio between the areas still constant?

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Latest reference of book is:

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