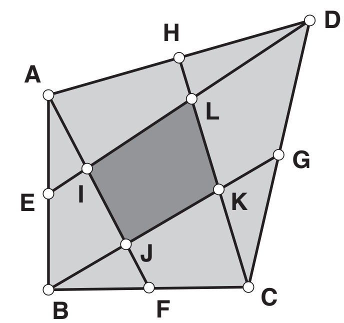
In this activity, you will compare the area of an entire quadrilateral to that of a smaller quadrilateral constructed within it.

CONJECTURE



- Open the sketch **Areas.gsp.**
- Find the ratio of the area of quadrilateral *ABCD* to the area of quadrilateral *IJKL*.

To find the ratio between two measurements, choose **Calculate** from the Number menu and then click on a measurement to enter it into the Calculator.

1. What do you notice about this ratio?

2. Drag any vertex of quadrilateral *ABCD* to a new position. Does your observation still hold?

3. Summarize your observations above by writing a conjecture.

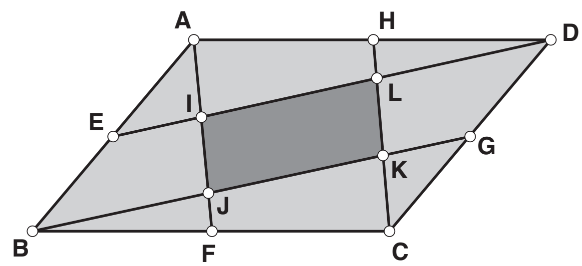
4. How certain are you that your conjecture is always true? Record your level of certainty on the number line and explain your choice.

0% 25% 50% 75% 100%

CHALLENGE If you believe your conjecture in Question 3 is always true, provide some examples to support your view and try to convince your partner or members of your group. Even better, support your conjecture with a logical explanation or a convincing proof. If you suspect your conjecture is not always true, try to supply counterexamples.

Repeat the previous investigation for a parallelogram.

- Open the sketch **Areas 2.gsp.**
- Again, find the ratio of the areas of the two quadrilaterals.



5. What do you now notice about this ratio?

To find the ratio between two measurements, choose **Calculate** from the Number menu and then click on a measurement to enter it into the Calculator.

6. Drag any of the vertices of parallelogram *ABCD* to a new position. Does your observation/conjecture still hold?

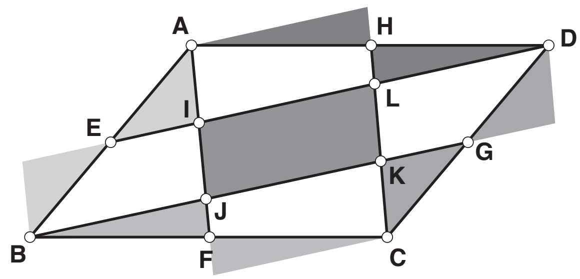
7. Formulate a conjecture based on your observations.

8. How certain are you that your conjecture is always true? Record your level of certainty on the number line and explain your choice.

CHALLENGE If you believe your conjecture is always true, provide some examples to support your view and try to convince your partner or members of your group. Even better, support your conjecture with a logical explanation or a convincing proof. If you suspect your conjecture or your partner's conjecture is not always true, try to supply counterexamples.

EXPLAINING

Press the button *Half turn triangles*. What do you observe? Use your observation to explain why your conjecture is true for a parallelogram.



AREAS (PAGE 73)

The purpose of this worksheet is to caution students not to make generalizations too quickly; they must be sure to explore many different variations, in particular looking at special or borderline cases. Students who don't test extreme cases can be led (or misled) to a false conjecture by the sketch.

At the beginning, check that the measurement and calculation accuracy in the Preferences is set to units (because students are then more likely to make the false conjecture). You can use this activity to introduce the verification (checking) function of proof. Point out that this activity illustrates that there are some cases in which it's difficult to really be sure that an empirical check has been sufficient.

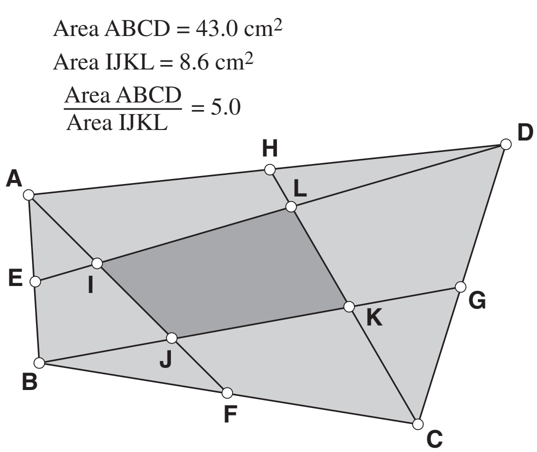
The second page of the activity is optional. Once students have discovered that their conjectures are not true in general, in the second part of the activity they discover a special case (a parallelogram) in which it is true.

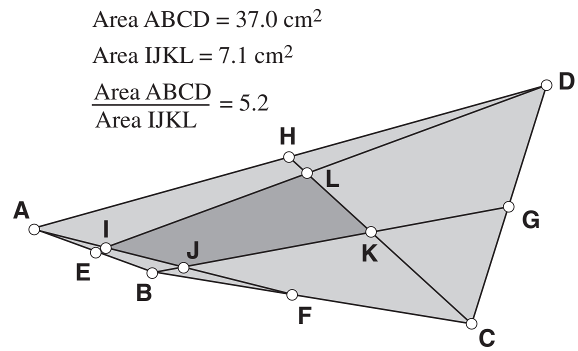
Prerequisites: None.

Sketches: Areas.gsp and **Areas 2.gsp.**

CONJECTURE

1. 5 : 1.
2. Probably yes.
3. Students will probably make the conjecture that the ratio of the given areas is always 5 : 1.
4. Answers will vary.





CHALLENGE With the ratio between the areas of the two quadrilaterals shown by $\frac{\text{Area } ABCD}{\text{Area } IJKL}$, it is hardly likely that students would come up with counterexamples. You could even suggest to your students to try more accurate measurement and calculation. Some students may, however, become bothered by their inability to construct an explanation and may begin to suspect that the result is not generally true.

5. 5 : 1.
6. Yes.
7. In a parallelogram *ABCD* with *E, F, G,* and *H* the respective midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ the ratio of the area of *ABCD* to the area of the quadrilateral formed by $\overline{AF}$, $\overline{BG}$, $\overline{CH}$, and $\overline{DE}$ is 5 : 1.
8. Answers will vary.

EXPLAINING

Since the figure as a whole has half-turn symmetry, *IJKL* has half-turn symmetry as well and is therefore also a parallelogram. The half-turns of the indicated triangles create four parallelograms, each congruent to *IJKL* and surrounding it. Therefore, Area *ABCD* : Area *IJKL* = 5 : 1.

An Update

The quadrilateral *IJKL* is called a *midvexogram* by Winicki-Landman (2001). The conjecture by Sylvie Penchaliah mentioned in the Acknowledgments, namely, that the ratio of the area of a (convex) quadrilateral to that of its midvexogram is always greater than or equal to 5 (also mentioned in Keyton 1997) was proven in 1999 by three mathematicians from the University of Kentucky—Avinash Sathaye, Carl Eberhart, and Don Coleman. Using the symbolic processing ability of Maple, they have also shown that this ratio is precisely 5 when the midvexogram is a trapezoid and that in all other cases the ratio is always less than 6 (although there are quadrilaterals for which this ratio can be as close to 6 as wanted). Their paper can be downloaded from http://www.ms.uky.edu/~carl/coleman/coleman2.html.

This proof, though convincing, is hardly explanatory, and the problem of finding a short, elegant, and explanatory geometric proof remains open.

VARIGNON AREA (PAGE 76)

This activity follows the Areas activity, and it is expected that students will be a bit more skeptical here about their Sketchpad observations and thus more motivated to seek additional verification or conviction. The focus of this activity is therefore on introducing the verification function of proof.

Prerequisites: The Kite Midpoints activity or knowledge of the result that the line connecting the midpoints of two sides of a triangle is parallel to the third side and half its length. Properties of parallelograms. Conditions for congruency.

Sketch: Varignon Area.gsp.

CONJECTURE

1. *EFGH* is a parallelogram. (This is true even for concave and crossed cases.)
2. The area of the parallelogram is half that of the original quadrilateral.
3. No.
4. No.
5. The midpoints of the sides of a quadrilateral form a parallelogram.
6. Responses will vary.

PROVING

7. $\overline{EF} \parallel \overline{AC} \parallel \overline{HG}$, since *E* and *F* are midpoints of sides *AB* and *CB* in triangle *ABC* and *H* and *G* are midpoints of sides *AD* and *CD* in triangle *ADC*.
8. $\overline{EH} \parallel \overline{BD} \parallel \overline{FG}$ (same reasons).
9. $\overline{EF} \parallel \overline{HG}$ and $\overline{EH} \parallel \overline{FG}$, so opposite sides are parallel, and therefore *EFGH* is a parallelogram. Another way of proving it is to note in Question 7 that not only is $\overline{EF} \parallel \overline{HG}$, but since both *EF* and *HG* are equal to half *AC,* they are also equal to each other. So one pair of opposite sides are equal and parallel, from which it follows that *EFGH* is a parallelogram.

   Note: You may also wish to ask your students to prove that the result is also true in the concave and crossed